Let $f: \mathbb{R} \to [0, \infty)$. The undergraph of f is
$$\mathcal{U}f = \{(x,y) \in \mathbb{R} \times [0,\infty) \mid 0 \leq y < f(x)\}$$

f is Lebesgue measurable if $\mathcal{U}f$ is measurable
If f is measurable, then the Lebesgue integral of f is the measure of the undergraph $\int f = m(\mathcal{U}f)$
f is Lesbegue integrable if it is measurable and its integral is finite

Monotone convergence theorem
Let (f_n) be a sequence of measurable functions $f_n: \mathbb{R} \to [0,\infty)$
If $f_n \uparrow f$ almost everywhere, then $\int f_n \uparrow \int f$

The completed undergraph of $f: \mathbb{R} \to [0,\infty)$ is
$$\hat{\mathcal{U}}f = \{(x,y) \in \mathbb{R} \times [0,\infty) \mid 0 \leq y \leq f(x)\}$$
i.e, the undergraph including the line

An affine linear transformation is a transformation which preserves co-linearity and proportion.
If $T: \mathbb{R}^n \to \mathbb{R}^n$ is an affine linear transformation (of the form $T(x) = A(x) + b$, where A is an $n \times n$ matrix and $b \in \mathbb{R}^n$) then $\forall E \in \mathbb{R}^n$, $m(T(E)) = |\det(A)| m(E) = |T| m(E)$

$\hat{\mathcal{U}}f$ is measurable iff $\mathcal{U}f$ is measurable.
If $\hat{\mathcal{U}}f$ and $\mathcal{U}f$ are measurable, then $m(\mathcal{U}f) = m(\hat{\mathcal{U}}f)$

If (f_n) is a sequence of integrable functions that converges monotonically down to a function f almost everywhere,
$$\int f_n \downarrow \int f$$

Let $f_n : X \to [0,\infty)$ be a sequence of functions
The lower and upper envelope sequences are
$\underline{f_n}(x) = \inf\{f_k(x) \mid k \geq n\}$ $\overline{f_n}(x) = \sup\{f_k(x) \mid k \geq n\}$

$\mathcal{U}(\overline{f_n}) = \bigcup_{k \geq n} \mathcal{U}(f_k)$ $\hat{\mathcal{U}}(\underline{f_n}) = \bigcap_{k \geq n} \hat{\mathcal{U}}(f_k)$

Dominated convergence theorem

If $f_n : \mathbb{R} \to [0,\infty)$ is a sequence of measurable functions s.t $f_n \to f$ almost everywhere, and $\exists\, g : \mathbb{R} \to [0,\infty)$ s.t $g(x) \geq f_n(x)$ $\forall x$ and $\int g < \infty$, then f is integrable, and
$\int f_n \to \int f$ $\mathcal{U}(f_n) \subset \mathcal{U}(g)$ $m(\mathcal{U}(f_n)) \leq m(\mathcal{U}(g)) < \infty$

The pointwise limit of measurable functions is measurable

Fatou's lemma:

If $f_n : \mathbb{R} \to [0,\infty)$ is a sequence of measurable functions
$\int \liminf f_n \leq \liminf \int f_n$

Let $f, g : \mathbb{R} \to [0,\infty)$ be measurable functions
a) If $f \leq g$ then $\int f \leq \int g$
b) If $\mathbb{R} = \bigsqcup_{k=1}^{\infty} X_k$ for measurable X_k then $\int f = \sum_{k=1}^{\infty} \int_{X_k} f$
c) If $X \subset \mathbb{R}$ is measurable then $mX = \int \chi_X$
d) If $m(X) = 0$ then $\int_X f = 0$
e) If $f(x) = g(x)$ almost everywhere then $\int f = \int g$
f) If $c \geq 0$ then $\int cf = c \int f$
g) $\int f = 0$ iff $f(x) = 0$ for almost ~~everywhere~~ every x
h) $\int f + g = \int f + \int g$

If $f: \mathbb{R} \to \mathbb{R}$ then the f-translation $T_f : \mathbb{R}^2 \to \mathbb{R}^2$ takes $(x, y) \mapsto (x, y + f(x))$

$T_f \circ T_g = T_{f+g} = T_g \circ T_f$

If $f: \mathbb{R} \to [0, \infty)$ is integrable then T_f preserves planar Lebesgue measure. i.e, T_f is a mesisometry

If $g, f: \mathbb{R} \to [0, \infty)$ are integrable then

$$\int f + g = \int f + \int g$$

If $f_k: \mathbb{R} \to [0, \infty)$ is a sequence of integrable functions,

$$\sum_{k=1}^{\infty} \int f_k = \int \sum_{k=1}^{\infty} f_k$$

The set of measurable functions $f: \mathbb{R} \to \mathbb{R}$ is a vector space, the set of integrable functions is a subspace, and integration is a linear map.

Let $\Omega \subset \mathbb{R}^n$ be measurable and let $f: \Omega \to \mathbb{R}$ be a measurable function.
f is a simple function if the image $f(\Omega)$ is finite
i.e $\exists$ a finite list $\{c_1, c_2, \dots, c_N\} \subset \mathbb{R}$ s.t $\forall x \in \Omega, f(x) = c_i$

Let $\Omega \subset \mathbb{R}^n$ be measurable and $f, g: \Omega \to \mathbb{R}$ be simple.
$f+g$ and cf $\forall c \in \mathbb{R}$ are also simple

Let $\Omega \subset \mathbb{R}^n$ be measurable and $f: \Omega \to \mathbb{R}$ be simple
$\exists\, c_1, c_2, \dots, c_N \in \mathbb{R}$ and disjoint measurable sets $E_1, E_2, \dots, E_N$
s.t $\quad f = \sum_{i=1}^{N} c_i \chi_{E_i}$

Let $\Omega \subset \mathbb{R}^n$, $f: \Omega \to \mathbb{R}$ be measurable, $f(x) \geq 0$ $\forall x \in \Omega$
$\exists$ a sequence of simple functions $f_n: \Omega \to \mathbb{R}$ s.t
$0 \leq f_1(x) \leq f_2(x) \leq \dots \quad \forall x \in \Omega$
and $\lim_{n \to \infty} f_n(x) = f(x)$ $\forall x \in \Omega$

Let $\Omega \subset \mathbb{R}^n$ be measurable and $f: \Omega \to \mathbb{R}$ be simple and non-negative. The lebesgue integral of f on Ω is defined

$$\int_\Omega f := \sum_{\lambda \in f(\Omega)} \lambda m(\{x \in \Omega \mid f(x) = \lambda\})$$

Let $\Omega \subset \mathbb{R}^n$ be measurable and let $E_1, \dots, E_N$ be a finite, disjoint list of measurable subsets of Ω. Let $c_1, \dots, c_N \geq 0$

$$\int_\Omega \sum_{j=1}^{N} c_j \chi_{E_j} = \sum_{j=1}^{n} c_j m(E_j)$$

Let Ω be measurable and $f, g: \Omega \to \mathbb{R}$ be non-negative simple functi

a) $0 \leq \int_\Omega f \leq \infty \qquad \int_\Omega f = 0$ iff $m(\{x \in \Omega \mid f(x) \neq 0\}) = 0$
b) $\int_\Omega f + g = \int_\Omega f + \int_\Omega g$
c) $\forall c > 0, \; \int_\Omega cf = c \int_\Omega f$
d) $f(x) \leq g(x)$ $\forall x \in \Omega \Rightarrow \int_\Omega f \leq \int_\Omega g$

Let $f, g : \Omega \to \mathbb{R}$. If $f(x) \geq g(x)$ $\forall x \in \Omega$, we say f majorises g, g minorises f, or f dominates g

Let $\Omega \subset \mathbb{R}^n$ be measurable. Let $f : \Omega \to [0, \infty]$ be measurable and non-negative. The Lebesgue integral of f on Ω is,

$$\int_\Omega f := \sup \left\{ \int_\Omega s \;\middle|\; s \text{ is simple, non-negative and minorises } f \right\}$$

Let Ω be measurable and $g, f : \Omega \to [0, \infty]$ be non-negative measurable functions

a) $0 \leq \int_\Omega f \leq \infty$ $\quad \int_\Omega f = 0$ iff $f(x) = 0$ for almost every $x \in \Omega$

b) $\forall c > 0$ $\quad \int_\Omega cf = c \int_\Omega f$

c) $f(x) \leq g(x)$ $\forall x \in \Omega \Rightarrow \int_\Omega f \leq \int_\Omega g$

d) $f(x) = g(x)$ for almost every $x \Rightarrow \int_\Omega f = \int_\Omega g$

e) $\Omega' \subseteq \Omega$ is measurable $\Rightarrow \int_{\Omega'} f = \int_\Omega f \chi_{\Omega'} \leq \int_\Omega f$

Lebesgue monotone convergence theorem

Let $\Omega \subset \mathbb{R}^n$ be measurable and let (f_n) be a sequence of non-negative measurable functions $f_n : \Omega \to \mathbb{R}$ s.t

$0 \leq f_1(x) \leq f_2(x) \leq \dots \leq \dots$ $\forall x \in \Omega$ Then,

$$0 \leq \int_\Omega f_1 \leq \int_\Omega f_2 \leq \dots \quad \text{and} \quad \int_\Omega \sup_n f_n = \sup_n \int_\Omega f_n$$

Let $\Omega \subset \mathbb{R}^n$ be measurable and $f, g : \Omega \to [0, \infty]$ be measurable.

$$\int_\Omega f + g = \int_\Omega f + \int_\Omega g$$

Let $\Omega \subset \mathbb{R}^n$ be measurable and $g_1, g_2, \dots$ be a sequence of non-negative measurable functions $g_i : \Omega \to [0, \infty]$

$$\int_\Omega \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int_\Omega g_n$$

Fatou's lemma:

Let $\Omega \subseteq \mathbb{R}^n$ be measurable and let $f_1, f_2, \dots$ be a sequence of nonnegative functions $f_i : \Omega \to [0, \infty]$

$$\int_\Omega \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_\Omega f_n$$

Let Ω be a measurable ~~subset~~ and let $f: \Omega \to [0,\infty]$ be a non-negative measurable function s.t $\int_\Omega f$ is finite. Then f is finite almost everywhere.

$$m(\{x \in \Omega \mid f(x) = +\infty\}) = 0$$

Borel-Cantelli lemma:

Let $\Omega_1, \Omega_2, \dots$ be measurable in $\mathbb{R}^n$ s.t $\sum_{n=1}^{\infty} m(\Omega_n)$ is finite. Then,

$$m(\{x \in \mathbb{R}^n \mid x \in \Omega_n \text{ for infinitely many } n\}) = 0$$

Almost every $x \in \mathbb{R}^n$ belongs to finitely many Ω_n.

Let $\Omega \subset \mathbb{R}^n$ be measurable. $f : \Omega \to \mathbb{R}^+$ is absolutely integrable if $\int_\Omega |f|$ is finite

If $f: \Omega \to \mathbb{R}$ is a function, we define $f^+, f^- : \Omega \to [0, \infty]$,

$$f^+ := \max(f, 0) \qquad f^- := -\min(f, 0)$$

If $f: \Omega \to \mathbb{R}$ is absolutely integrable, the Lesbesgue integral,

$$\int_\Omega f := \int_\Omega f^+ - \int_\Omega f^-$$

If f is non-negative then $f^+ = f$ and $f^- = 0$

$$\left|\int_\Omega f\right| \leq \int_\Omega f^+ + \int_\Omega f^- = \int_\Omega |f|$$

Let Ω be measurable $f, g : \Omega \to \mathbb{R}$ be absolutely integrable

a) $\forall c \in \mathbb{R}$, cf is absolutely integrable, $\int_\Omega cf = c\int_\Omega f$

b) $f+g$ is absolutely integrable $\int_\Omega f+g = \int_\Omega f + \int_\Omega g$

c) $f(x) \leq g(x)\ \forall x \in \Omega \Rightarrow \int_\Omega f \leq \int_\Omega g$

d) $f(x) = g(x)$ for ~~every~~ almost every $x \in \Omega \Rightarrow \int_\Omega f = \int_\Omega g$

Let $\Omega \subset \mathbb{R}^n$ be measurable, $f_1, f_2, \ldots : \Omega \to \mathbb{R}$ be a sequence of measurable functions converging pointwise.
Let $F : \Omega \to [0, \infty]$ s.t $|f_n(x)| \leq F(x)\ \forall x \in \Omega, n \in \mathbb{N}$ and $F(x)$ is absolutely integrable

$$\int_\Omega \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Omega f_n$$

Let $\Omega \subset \mathbb{R}^n$ be measurable, $f : \Omega \to \mathbb{R}$

The upper Lebesgue integral $\overline{\int}_\Omega f$ is defined,

$\overline{\int}_\Omega f :=$ ~~sup~~ $\{\int_\Omega g \mid g$ is absotulley integrabland majorises $f\}$

The lower Lebesgue integral $\underline{\int}_\Omega f$ is defined

$\underline{\int}_\Omega f := \sup\{\int_\Omega g \mid g$ is absolutely integrable and minorises $f\}$

Let $\Omega \subset \mathbb{R}^n$ be measurable $f : \Omega \to \mathbb{R}$. Let $\overline{\int}_\Omega f = \underline{\int}_\Omega f$

$\int_\Omega f = \underline{\int}_\Omega f = \overline{\int} f$ and f is absolutely integrable

Let $I \subseteq \mathbb{R}$ be an interval, $f: I \to \mathbb{R}$ be a Reimann integrable function. Then f is absolutely integrable, and

$$\int_I f = R.\int_I f$$

Fubini's theorem

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be absolutely integrable

$\exists$ absolutely integrable functions $F, G: \mathbb{R} \to \mathbb{R}$ s.t

for almost every x, $f(x,y)$ is absolutely integrable in y,

$$F(x) = \int_{\mathbb{R}} f(x,y)\,dy$$

and for almost every y, $f(x,y)$ is absolutely integrable in x,

$$G(\cancel{x}y) = \int_{\mathbb{R}} f(x,y)\,dx$$

and $$\int_{\mathbb{R}} F(x)\,dx = \int_{\mathbb{R}^2} f = \int_{\mathbb{R}} G(y)\,dy$$