Lemma 0: Let $E, F \subset \mathbb{R}^d$ s.t $\inf\{|x-y| \mid x \in E, y \in F\} > 0$

Finite subadditivity $\Rightarrow m^*(E \cup F) \leq m^*(E) + m^*(F)$

Let $\varepsilon > 0$. $\exists$ a collection of open boxes $(B_i)_{i \in J}$ s.t $\sum_{i \in J} Vol(B_i) < m^*(E \cup F) + \varepsilon$ (where J is countable)

Given that $dist(E, F) > 0$ we can partition $(B_i)_{i \in J}$ into two collections of smaller boxes (P_i) and (Q_i) s.t (P_i) covers E, (Q_i) covers F and,

$(\bigcup P_i) \cap F = (\bigcup Q_i) \cap E = \emptyset$

$(P_i) \cup (Q_i) \subseteq (B_i) \Rightarrow \sum_i Vol(P_i) + \sum Vol(Q_i) \leq \sum Vol(B_i)$

$m^*(E) \leq \sum Vol(P_i)$ and $m^*(F) \leq \sum Vol(Q_i)$

$\Rightarrow m^*(E) + m^*(F) \leq \sum Vol(B_i) < m^*(E \cup F) + \varepsilon$

$\Rightarrow m^*(E) + m^*(F) < m^*(E \cup F) + \varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow m^*(E) + m^*(F) \leq m^*(E \cup F)$ and $m^*(E) + m^*(F) \geq m^*(E \cup F)$

$\therefore m^*(E) + m^*(F) = m^*(E \cup F)$ as required

Lemma 1: Let $A \subseteq \mathbb{R}^n$

Let $(B_i)_{i \in J}$ be a countable collection of open boxes covering A

Let $U = \bigcup_{i \in J} (B_i)$. U is an open set covering A

$\Rightarrow \inf\{m^*(U) \mid A \subset U, U \text{ is open}\} \leq m^*(U)$

$m^*(U) = m^*(\bigcup_{i \in J} (B_i)) \leq \sum_{i \in J} m^*(B_i) \leq \sum_{i \in J} Vol(B_i)$

$\therefore \forall (B_i)_{i \in J} \; \exists U$ s.t $m^*(U) \leq \sum_{i \in J} Vol(B_i)$

$\Rightarrow \inf\{m^*(U) \mid A \subset U, U \text{ is open}\} \leq \inf\{\sum Vol(B_i) \mid (B_i) \text{ covers } A\}$

$\Rightarrow \inf\{m^*(U) \mid A \subset U, U \text{ is open}\} \leq m^*(A)$

$\forall U \supset A$ monotonicity implies $m^*(A) \leq m^*(U)$

$\Rightarrow m^*(A) \leq \inf\{m^*(U) \mid A \subset U, U \text{ is open}\}$

$\therefore m^*(A) = \inf\{m^*(U) \mid A \subset U, U \text{ is open}\}$ as required

Lemma 3: Let $A \subset \mathbb{R}^n$ be closed. Let $x \in A$.

Let $A_n = \{a \in A \mid d(a,x) \leq n\}$ $\forall n \in \mathbb{N}$

$A = \bigcup_{n \in \mathbb{N}} A_n$ where A_n is closed & bounded $\forall n \in \mathbb{N}$

RTP: A_n is measurable

A_n is bounded $\Rightarrow$ $\exists$ an open set U s.t $A_n \subseteq U$

From lemma 1, $m^*(A_n) = \inf(m^*(U))$

$\Rightarrow \forall \varepsilon > 0$ $\exists U$ s.t $m^*(U) < m^*(A_n) + \varepsilon$

Lemma 4: Let E be measurable

$\Rightarrow \forall \varepsilon > 0 \ \exists$ an open set U s.t $E \subset U$ and $m^*(U \setminus E) < \varepsilon$

$\Rightarrow \forall n \in \mathbb{N} \ \exists$ an open set U_n s.t $E \subset U_n$ and $m^*(U_n \setminus E) < \frac{1}{n}$

Let $U^c = \bigcup_{n \in \mathbb{N}} U_n^c$

U_n is open $\forall n \in \mathbb{N} \Rightarrow U_n^c$ is closed $\forall n \in \mathbb{N}$

$\therefore U^c$ is a countable union of closed sets

Lemma 2 & 3 $\Rightarrow U^c$ is measurable

$E^c = U^c \cup (E^c \setminus U^c)$ (as $U^c \subseteq E^c$)

$E^c \setminus U^c = (\bigcap_{n \in \mathbb{N}} U_n) \setminus E = \bigcap_{n \in \mathbb{N}} (U_n \setminus E)$

$\cancel{\forall N \in \mathbb{N} \ \bigcap_{n \in \mathbb{N}} \ \bigcap_{n=1}^{N} (U_n \setminus E) = (U_N \setminus E) \subseteq (U_{N+1} \setminus}$

$\forall N \in \mathbb{N}, \ \bigcap_{n=1}^{N} (U_n \setminus E) = (U_N \setminus E) \supseteq (U_{N+1} \setminus E) = \bigcap_{n=1}^{N+1} (U_n \setminus E)$

$\therefore$ by monotonicity $m^*(U_{n+1} \setminus E) \leq m^*(U_n \setminus E) \ \forall n \in \mathbb{N}$

$\Rightarrow m^*(\bigcap_{n \in \mathbb{N}} (U_n \setminus E)) \leq m^*(U_N \setminus E) \ \forall N \in \mathbb{N}$

$\Rightarrow m^*(\bigcap_{n \in \mathbb{N}} (U_n \setminus E)) \cancel{\leq} < \frac{1}{N} \ \forall N \in \mathbb{N}$

$\Rightarrow m^*(E^c \setminus U^c) < \frac{1}{N} \ \forall N \in \mathbb{N}$

$\Rightarrow m^*(E^c \setminus U^c) = 0$

$\therefore E^c = U^c \cup (E^c \setminus U^c)$ where U^c is measurable and $m^*(E^c \setminus U^c) = 0$

$\therefore E^c$ is measurable.