Homework 7

1) Let f and g be measurable
Let f^2 and g^2 be integrable
$\Rightarrow$ fg is measurable and f^2+g^2 is integrable
Let $fg^+ = \max(fg, 0)$ $\quad fg^- = -\min(fg, 0)$
$\forall x \in \mathbb{R}$ $\cancel{f(x)}g(x) = fg^+(x) - fg^-(x)$
$\forall x \in \mathbb{R}$ $f^2(x) + g^2(x) \geq \cancel{2 + fg(x)} + 2|f(x)g(x)|$
$\Rightarrow$ $f^2(x) + g^2(x) \geq fg^+(x) \geq 0$
$f^2(x) + g^2(x) \geq fg^-(x) \geq 0$
$\Rightarrow$ $\int fg^+ \cancel{(x)} < \int f^2 + g^2$
$\int fg^- \cancel{x} < \int f^2 + g^2$
$\Rightarrow$ $\cancel{\int} fg^+$ and fg^- are integrable
$\Rightarrow$ $fg = fg^+ - fg^-$ is integrable

Consider the vector space V of measurable functions on $\mathbb{R}$
$fg \in V$
Define the inner product $\langle u, v \rangle = \int uv$
$\langle v, v \rangle = \int v^2 \geq 0$ (definiteness)
$\langle u+v, w \rangle = \int (u+v)w = \int uw + \int vw = \langle u, w \rangle + \langle v, w \rangle$
$\langle v, v \rangle = 0$ ~~iff~~ $\Leftrightarrow \int v^2 = 0 \Leftrightarrow v = 0$
$\langle cv, u \rangle = \int cvu = c\int vu = c\langle v, u \rangle$
$\langle v, u \rangle = \int vu = \int uv = \langle u, v \rangle$
By the Cauchy Schwarz inequality:
$|\langle f, g \rangle| \leq \|f\| \, \|g\|$
$|\langle f, g \rangle| \leq \sqrt{\langle f, f \rangle} \sqrt{\langle g, g \rangle}$
$\int fg \leq \sqrt{\int f^2} \sqrt{\int g^2}$ as required

3) $f(x,y) = \begin{cases} \frac{1}{y^2} & \text{if } 0<x<y<1 \\ \frac{-1}{x^2} & \text{if } 0<y<x<1 \\ 0 & \text{otherwise} \end{cases}$

$\int f(x,y)\,dx = \int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx = \left[\frac{1}{y^2}x\right]_0^y + \left[\frac{1}{x}\right]_y^1$
$= \frac{1}{y} + 1 - \frac{1}{y} = 1$ which is finite

$\int f(x,y)\,dy = \int_0^x -\frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dy = \left[-\frac{1}{x^2}y\right]_0^x + \left[-\frac{1}{y}\right]_x^1$
$= -\frac{1}{x} - 1 + \frac{1}{x} = -1$ which is finite

$\cancel{\iint f\,dA} = \int_0^1 \int_0^y \frac{1}{y^2}\,dx\,dy = \int_0^1 \frac{1}{y}\,dy = [\ln y]_0^1 = \infty$
$\therefore$ the upper integral of f does not exist
$\Rightarrow f$ is not integrable

Corollary 43 applies only to functions $f: \mathbb{R} \to [0,\infty)$

4) a) By applying the Vitali covering lemma for closed balls to the proof of the Lebesgue density theorem, we can show that it holds for balanced density points

b)