Math 105, Homework 2

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Taught in Fall 2021.

Note: I proved Lemma 1 before proving Lemma 0, and used Lemma 1 in my proof of Lemma 0.

Lemma 0:

Claim. *Let $E, F \subseteq \mathbb{R}^n$ with dist$(E, F) > 0$. Then $m^*(E \cup F) = m^*(E) + m^*(F)$.*

Proof. (We assume Lemma 1.) Since $\text{dist}(E, F) > 0$, there are open sets A, B such that $E \subseteq A$, $F \subseteq B$, and $A \cap B = \emptyset$. For all such open sets small enough, the finite additivity of outer measure gives $m^*(A \cup B) = m^*(A) + m^*(B)$. Note that if A, B are open sets covering E, F respectively, then $A \cup B$ covers $E \cup F$, and if C is an open set covering $E \cup F$, then C covers both E and F individually as well. Hence the collection of open sets covering $E \cup F$ is precisely the collection of unions of any open set covering E with any open set containing F. Thus combining the above with Lemma 1 we get

$$\begin{aligned} m^*(E \cup F) &= \inf \{m^*(U) \mid U \text{ open and } E \cup F \subseteq U\} \\ &= \inf \{m^*(U) \mid U \text{ open and } E \subseteq U\} + \inf \{m^*(U) \mid U \text{ open and } F \subseteq U\} \\ &= m^*(E) + m^*(F). \end{aligned}$$

□

Lemma 1:

Claim. *Let $A \subseteq \mathbb{R}^n$. Then $m^*(A) = \inf \{m^*(U) \mid U$ open and $A \subseteq U\}$.*

Proof. Let $A \subseteq \mathbb{R}^n$. Then

$$\begin{aligned} m^*(A) &= \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ covers } A \right\} \\ &= \inf \left\{ \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ covers } U \right\} \mid A \subseteq U \text{ and } U \text{ open } \right\}. \end{aligned}$$

Now if U is an open set, then U is a countable union of open boxes $\{V_k^U\}$ and therefore $m^*(U) = \sum |V_k^U|$. then $m^*(A) \leq \sum |V_k^U|$ for all open sets U with $A \subseteq U$. Then

$$\begin{aligned} m^*(A) &\leq \inf \left\{ \sum |V_k^U| \mid U \text{ open and } A \subseteq U \right\} \\ &= \inf \{m^*(U) \mid U \text{ open and } A \subseteq U\}. \end{aligned}$$

We now want to show $m^*(A) \geq \inf \{m^*(U) \mid U \text{ open and } A \subseteq U\}$.

Suppose for contradiction that $m^*(A) < \inf \{m^*(U) \mid U \text{ open and } A \subseteq U\}$. Then

$$\inf \left\{ \sum |B_k| \mid \{B_k\} \text{ covers } A \right\} < \inf \{m^*(U) \mid U \text{ open and } A \subseteq U\}$$

which implies there exists a covering $\{B_k^A\}$ of A such that $\sum |B_k^A| < m^*(U)$ for all U open with $A \subseteq U$. Since for each U open such that $A \subseteq U$ there exists $\epsilon_U > 0$ such that $\sum |B_k^A| = m^*(U) + \epsilon_U$, we have $m^*(U \setminus \bigcup_k B_k^A) \geq \epsilon_U > 0$. Then for each k, there exists $\delta_k > 0$ such that where $B_k^A = \prod_{i=1}^n (a_i, b_i)$ we have

$$B_k' = \prod_{i=1}^n (a_i - \delta_k, b_i + \delta_k)$$

with

$$A \subseteq \bigcup_k B_k^A \subsetneq \bigcup_k B_k' \subsetneq U$$

and therefore

$$m^*(A) \leq \sum |B_k^A| < \sum |B_k'| < m^*(U).$$

But since $\bigcup_k B_k'$ is an open set, by the last inequality in the chain above we have

$$\sum |B_k'| < m^*(\bigcup_k B_k') = \sum |B_k'|, \tag{1}$$

which is a contradiction. □

Lemma 2:

Claim. *If $\{E_i\}$ is a countable collection of measurable sets, then $\bigcup_i E_i$ is measurable.*

Proof. For each i, and for each $\epsilon > 0$, there exists U_i such that U_i is open with $E_i \subseteq U_i$, and $m^*(U_i \setminus E_i) < \epsilon/2^i$. Let $U = \bigcup_i U_i$ and $E = \bigcup_i E_i$. Then $U \setminus E = \bigcup_i (U_i \setminus E_i)$ (Prove this! I'm confident it's true, but haven't made my argument rigorous.) and we have

$$m^*(U \setminus E) = m^*(\bigcup_i (U_i \setminus E_i)) \leq \sum_i m^*(U_i \setminus E_i) < \sum_i \epsilon/2^i = \epsilon$$

and therefore E is measurable. □

Lemma 3:

Claim. *Every closed subset of $\mathbb{R}^n$ is measurable.*

Proof. If F is closed, then $F = G \cup \partial G$ where ∂G is the boundary of F (i.e. $\partial G = \text{cl}(G) \cap \text{cl}(G^c)$, where cl() is the topological closure operation). G, being an open set, can be written as the countable union of open sets $\{G_i\}$, which in turn can be written as the countable union of open balls $B(x_i, \epsilon_i)$ centered at x_i with radius ϵ_i for some countable collection of points $x_i \in \mathbb{R}^n$ and values $\epsilon_i > 0$. So

$$F = \left(\bigcup_i B(x_i, \epsilon_i) \right) \cup \partial G.$$

We'll show that $\partial G = \bigcup_i (\partial G_i)$.

$x \in \partial G$ if and only if $x \notin G$ and for each open ball $B(x, \epsilon)$ there exists $g \in G$ such that $g \in B(x, \epsilon)$. This is true if and only if $x \notin G_i$ for each i and for each open ball $B(x, \epsilon)$ there exist $g_j \in G_j$ for some j such that $g_j \in B(x, \epsilon)$. This is true if and only if $x \in \partial G_i$ for some i. Thus $\partial G = \bigcup_i \partial G_i$.

Then

$$\begin{aligned} F &= \left(\bigcup_i G_i\right) \cup \left(\bigcup_i \partial G_i\right) \\ &= \bigcup_i (G_i \cup \partial G_i) \\ &= \bigcup_i \mathrm{cl}(G_i). \end{aligned}$$

$\mathrm{cl}(G_i)$ is bounded if we take $\{G_i\}$ to be one of the collections comprised of a countable number of open balls, as each open ball G_i is bounded. Then $\mathrm{cl}(G_i)$ is a closed and bounded, and therefore compact, subset of $\mathbb{R}^n$.

We now just need to show that any compact subset of $\mathbb{R}^n$ is measurable.

Let $X \subseteq \mathbb{R}^n$ be compact. Since X is compact, there are closed n intervals $\{[a_i, b_i]\}$ such that $\prod_{i=1}^n [a_i, b_i]$ contains X. Then for each $\epsilon > 0$, we have an open set U of the form

$$U = \prod_{i=1}^n (a_i - \epsilon/(2n), b_i + \epsilon/(2n))$$

so that $X \subsetneq U$. Let $R = \mathbb{Q}^n \cap U$, and well-order R by $\prec$. Let q_i be the i'th term in the well ordering $(R, \prec)$. For each i, let $q_{i,j}$ denote the j'th coordinate of q_i and let

$$U_i = \prod_{j=1}^n (q_{i,j} - \epsilon/(n2^{i+1}), q_{i,j} + \epsilon/(n2^{i+1})). \tag{2}$$

The collection $\{U_i\}$ is a countable open covering of X and hence can be reduced to a finite subcover $\{U_{i,1}, \ldots, U_{i,k}\}$ and

$$\sum_{j=1}^k |U_{i,j}| < \sum_{j=1}^\infty |U_j| = \sum_{j=1}^\infty \epsilon/2^j = \epsilon.$$

Let $\mathcal{U} = \bigcup_{j=1}^k U_{i,j}$. Since $X \subseteq \mathcal{U}$ and so is $\mathcal{U} \setminus X$, finite sub-additivity and teh above inequality gives

$$m^*(\mathcal{U} \setminus X) \leq m^*(\mathcal{U}) < \epsilon.$$

$\square$

Lemma 4:

Claim. *If E is measurable, then E^c is measurable.*

Proof. Suppose E is measurable. Then for each $\epsilon > 0$ there exists an open set U_ϵ such that $E \subseteq U_\epsilon$ and $m^*(U_\epsilon \setminus E) < \epsilon$. For each ϵ, let $G_\epsilon = (U_\epsilon^c) \cup (U_\epsilon \setminus E) = E^c$ (where A^c denotes the complement of A for any set A).

U_ϵ^c is closed since U_ϵ is open. Their boundary, ∂U_ϵ is defined to be

$$\partial U_\epsilon = \mathrm{cl}(U_\epsilon^c) \cap \mathrm{cl}(U_\epsilon) = U_\epsilon^c \cap \mathrm{cl}(U_\epsilon)$$

where cl is the closure operation for usual topology on $\mathbb{R}^n$ induced by the usual metric on $\mathbb{R}^n$, and where the last equality holds because $U_\epsilon^c = \mathrm{cl}(U_\epsilon^c)$ since U_ϵ^c is closed.

Let $\mathscr{C} = \{U_\epsilon^c\}$ and $\Delta = \bigcap_\epsilon \partial U_\epsilon^c$. Together, $\left(\bigcup_{C \in \mathscr{C}} C\right) \cup \Delta = E^c$. □

Now I need to prove that Δ has measure zero.

∂A has empty interior for any set A and hence contains no open interval. We have $\mathbb{R}^n = A \cup (A^c)^\circ \cup \partial A$ for any subset A of $\mathbb{R}^n$ where A° is the interior of A. Since $\mathbb{R}^n$ is closed, if A is open, then since A° is open as well, it follows that ∂A is closed. Then Lemma 3 implies that Δ is measurable. Since $\{U_{1/k}^c\} \mid k \in \mathbb{N}\} \subseteq \mathscr{C}$ is a countable collection of open sets, we have $\bigcup_{k \in \mathbb{N}} U_{1/k}^c$ is open and thus measurable, we have E^c being the union of two measurable sets:

$$E^c = \bigcup_{k \in \mathbb{N}} U_{1/k}^c \cup \Delta$$

it follows that E^c is measurable.