

# Math 105, Homework 1

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Taught in Fall 2021.

## Tao, Analysis II, Lemma 7.2.5

### Empty Set

**Claim.**  $m^*(\emptyset) = 0$ .

*Proof.* In  $\mathbb{R}^n$ , the open box  $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i < 1/\epsilon^n\}$  covers  $\emptyset$  for each  $\epsilon > 0$ . Then  $m^*(\emptyset) \leq \epsilon$  for each  $\epsilon > 0$ , and so by positivity (proven next) we have  $m^*(\emptyset) = 0$ .  $\square$

### Positivity

**Claim.**  $0 \leq m^*(\Omega) \leq +\infty$  for every measurable set  $\Omega$ .

*Proof.* The upper bound of the inequality is immediate. For the lower bound, for any  $\Omega$ ,  $m^*(\Omega)$  is the infimum over a set of sums of nonnegative numbers, i.e. the infimum over a set of nonnegative numbers. Clearly, the infimum of such a set is nonnegative.  $\square$

### Monotonicity

**Claim.** If  $A \subseteq B \subseteq \mathbb{R}^n$ , then  $m^*(A) \leq m^*(B)$ .

*Proof.* If  $\{B_k\}$  is an open box covering of  $B$ , then it covers  $A$  as well. Hence the sum of the volumes of the  $B_k$ 's in the covering is greater than or equal to the infimum over the sum of volumes of the open box coverings of  $A$ .  $\square$

### Finite Sub-Additivity

**Claim.** If  $(A_j)_{j \in J}$  is a finite collection of subsets of  $\mathbb{R}^n$ , then  $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$ .

*Proof.* Let

$$\begin{aligned} A &= \bigcup_{j \in J} A_j, \\ W_j &= \left\{ \sum_k |B_k| \mid \{B_k\} \text{ is a countable open box cover of } A_j \right\} \text{ for } j \in J, \\ X &= \left\{ \sum_k |B_k| \mid \{B_k\} \text{ is a countable open box cover of } A \right\}, \text{ and} \\ Y &= \left\{ \sum_j \sum_k |B_{j,k}| \mid \text{For some } j \in J, \{B_{j,k}\} \text{ is a countable open box cover of } A_j \right\}. \end{aligned}$$

Then  $X \subseteq Y$ . We thus have

$$\begin{aligned}
 m^*(A) &= \inf X \\
 &\leq \inf Y \\
 &= \sum_{j \in J} \inf W_j \\
 &= \sum_{j \in J} m^*(A_j).
 \end{aligned}$$

□

### Countable Sub-Additivity

**Claim.** Suppose  $(A_j)_{j \in J}$  is a countable collection of subsets of  $\mathbb{R}^n$ . Then  $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$ .

*Proof.* Suppose  $(A_j)_{j \in J}$  is a countable collection of subsets of  $\mathbb{R}^n$ . We can well order the collection, denoted now by  $(A_m)_{m \in \mathbb{N}}$ . Let  $A = \bigcup_{j \in J} A_j$ . We have

$$\begin{aligned}
 m^*(A) &= \inf \left\{ \sum_k |B_k| \mid \{B_k\} \text{ is a countable open box cover of } A \right\} \\
 &\leq \sum_{i=1}^p m^*(A_i) \text{ for } p \in \mathbb{N} \\
 &= \sum_{i=1}^p \inf \left\{ \sum_k |B_k| \mid \{B_k\} \text{ is a countable open box cover of } A_i \right\} \\
 &\leq \sum_{i=1}^{\infty} \inf \left\{ \sum_k |B_k| \mid \{B_k\} \text{ is a countable open box cover of } A_i \right\} \\
 &= \sum_{j \in J} m^*(A_j)
 \end{aligned}$$

□

### Translation Invariance

**Claim.** Let  $\Omega \subseteq \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ . Then  $m^*(x + \Omega) = m^*(\Omega)$ .

*Proof.* If  $\{B_k\}$  is a countable open box cover of  $\Omega$ , then  $\{x + B_k\}$  is a countable open box cover of  $x + \Omega$ . Given any  $B_k$  in the countable open box cover of  $\Omega$ , with volume  $|B_k|$ ,  $x + B_k$  has the same

volume. Thus the sets

$$\begin{aligned} W &= \left\{ \sum_k |B_k| \mid \{B_k\} \text{ is a countable open box cover of } \Omega \right\}, \\ X &= \left\{ \sum_k |x + B_k| \mid \{B_k\} \text{ is a countable open box cover of } \Omega \right\}, \text{ and} \\ Y &= \left\{ \sum_k |D_k| \mid \{D_k\} \text{ is a countable open box cover of } x + \Omega \right\} \end{aligned}$$

are all equal and we get  $m^*(\Omega) = \inf W = \inf Y = m^*(x + \Omega)$ .  $\square$

## Tao Exercise 7.2.2

**Claim.** Let  $A \subseteq \mathbb{R}^n$ ,  $B \subseteq \mathbb{R}^m$ . Then  $m_{n+m}^*(AB) \leq m_n^*(A)m_m^*(B)$ .

*Proof.* For convenience, we'll use the abbreviation COBC, which will stand for "countable open-box cover." By definition, we have

$$m_{n+m}^*(A \times B) = \inf \left\{ \sum |X_k| \mid \{X_k\} \text{ is a COBC of } A \times B \right\}$$

For each COBC  $\mathcal{X} = \{X_k\}$  of  $A \times B$ , for each  $X_k \in \mathcal{X}$ , we can write  $X_k = C_{A,k} \times C_{B,k}$  where  $C_{A,k} \subseteq \mathbb{R}^n$  and  $C_{B,k} \subseteq \mathbb{R}^m$ . For each  $X_k \in \mathcal{X}$ ,  $X_k$  is of the form

$$X_k = \prod_{i=1}^n I_i \times \prod_{j=1}^m J_j$$

where for each  $i, j$  with  $1 \leq i \leq n$  and  $1 \leq j \leq m$ ,  $I_i$  and  $J_j$  are open intervals in  $\mathbb{R}$ . Clearly  $\prod_{i=1}^n I_i = C_{A,k}$  and  $\prod_{j=1}^m J_j = C_{B,k}$ , meaning both  $C_{A,k}$  and  $C_{B,k}$  are open boxes in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  respectively, with  $|X_k| = |C_{A,k}||C_{B,k}|$ . Then we have

$$m_{n+m}^*(A \times B) = \inf \left\{ \sum |C_{A,k}||C_{B,k}| \mid \{X_k\} \text{ is a COBC of } A \times B \text{ and for each } k \text{ we have } X_k = C_{A,k} \times C_{B,k} \right\}$$

Since we have the inequality

$$\sum |C_{A,k}||C_{B,k}| \leq \left( \sum |C_{A,k}| \right) \left( \sum |C_{B,k}| \right)$$

it follows that

$$\inf_{\mathcal{X} \in \mathcal{C}} \left\{ \sum |C_{A,k}||C_{B,k}| \right\} \leq \inf_{\mathcal{X} \in \mathcal{C}} \left\{ \left( \sum |C_{A,k}| \right) \left( \sum |C_{B,k}| \right) \right\}$$

where  $\mathcal{C}$  is the collection of all COBCs of  $A \times B$ . The left hand side of the above equation is  $m_{n+m}^*(A \times B)$ .

Now, if  $\mathcal{X} = \{X_k\}$  is a COBC of  $A \times B$ ,  $\{C_{A,k}\}$  is a COBC of  $A$ , and  $\{C_{B,k}\}$  is a COBC of  $B$ . Conversely, if  $\{A_k\}$  is a COBC of  $A$ , then  $\{A_k \times \mathbb{R}^m\}$  is a COBC of  $A \times B$ , and if  $\{B_k\}$  is a COBC of  $B$ , then  $\{\mathbb{R}^n \times B_k\}$  is a COBC of  $A \times B$ . Hence  $\{A_k\}$  is a COBC of  $A$  if and only if  $\{A_k\} = \{C_{A,k}\}$  for some COBC  $\{X_k\}$  of  $A \times B$ , and  $\{B_k\}$  is a COBC of  $B$  if and only if  $\{B_k\} = \{C_{B,k}\}$  for some COBC  $\{X_k\}$  of  $A \times B$ . Then if  $\mathcal{A}$  and  $\mathcal{B}$  are the collections of all COBCs of  $A$  and  $B$  respectively,

$$\inf_{\mathcal{X} \in \mathcal{C}} \left\{ \sum |C_{A,k}| \right\} = \inf_{\{A_k\} \in \mathcal{A}} \left\{ \sum |A_k| \right\} = m_n^*(A)$$

and

$$\inf_{\mathcal{X} \in \mathcal{C}} \left\{ \sum |C_{B,k}| \right\} = \inf_{\{B_k\} \in \mathcal{B}} \left\{ \sum |B_k| \right\} = m_m^*(B)$$

and we therefore have

$$m_{n+m}^*(A \times B) \leq m_n^*(A)m_m^*(B)$$

□

### Tao Exercise 7.2.3, Part (a)

**Claim.** Let  $(A_j)_{j \in \mathbb{N}}$  be a sequence of increasing measurable subsets of  $\mathbb{R}^n$ , where  $\mathbb{R}^n$  is equipped with an arbitrary measure  $m$ . Then

$$m \left( \bigcup_{j=1}^{\infty} A_j \right) = \lim_{j \rightarrow \infty} m(A_j).$$

*Proof.*

**Claim** (Subclaim 1.).  $\lim_{j \rightarrow \infty} m(A_j) \leq m(A)$ .

*Proof.* For any  $j$ , any COBC of  $A$  is a COBC of  $A_j$ . Also, for any  $x, j$  with  $j \leq x$ , a COBC of  $A_x$  is a COBC of  $A_j$ . It follows that

$$\begin{aligned} m(A_j) &= \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ is a COBC of } A_j \right\} \\ &\leq \inf \left\{ \sum |B_k| \mid x > j \text{ and } \{B_k\} \text{ is a COBC of } A_x \right\} = m(A_x) \quad (j \leq x) \\ &\leq \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ is a COBC of } A \right\} = m(A). \end{aligned}$$

Since the sequence  $(m(A_j))_{j \in \mathbb{N}}$  is monotonically increasing and bounded above by  $m(A)$ , we have  $\lim_{j \rightarrow \infty} m(A_j) \leq m(A)$ . □

**Claim** (Subclaim 2.).  $m(A) \leq \lim_{j \rightarrow \infty} m(A_j)$

*Proof.* Suppose  $m(A) > \lim_{j \rightarrow \infty} m(A_j)$ . Then  $m(A) = \lim_{j \rightarrow \infty} m(A_j) + \epsilon$  for some  $\epsilon > 0$ . That is, for each  $j$  there exists some  $\epsilon_j > 0$  such that

$$\inf \left\{ \sum |B_k| \mid \{B_k\} \text{ is a COBC of } A \right\} = \inf \left\{ \sum |B_{j,k}| \mid \{B_{j,k}\} \text{ is a COBC of } A_j \right\} + \epsilon_j.$$

Since every COBC of  $A$  is a COBC of  $A_j$  for each  $j$ , there is a subset  $\alpha$  of  $A$  with nontrivial measure disjoint from each  $A_j$ . In particular, for each  $j$ , there exists a COBC  $\mathcal{C}_j$  of  $A_j$  with  $\alpha \cap \bigcup_{C \in \mathcal{C}_j} C = \emptyset$ . But  $A_j \subseteq \bigcup_{C \in \mathcal{C}_j} C$  hence  $A_j \cap \alpha = \emptyset$  for each  $j$ . Then  $\bigcup_{j=1}^{\infty} A_j \subsetneq A$ , which is a contradiction.  $\square$

We've shown that  $m(A) \leq \lim_{j \rightarrow \infty} m(A_j) \leq m(A)$ , and therefore  $m(A) = \lim_{j \rightarrow \infty} m(A_j)$ .  $\square$

### Tao Exercise 7.2.3, Part (B)

**Claim.** Let  $(A_j)_{j \in \mathbb{N}}$  be a sequence of decreasing measurable subsets of  $\mathbb{R}^n$ , where  $\mathbb{R}^n$  is equipped with an arbitrary measure  $m$ , and  $m(A_1) < +\infty$ . Then

$$m \left( \bigcap_{j=1}^{\infty} A_j \right) = \lim_{j \rightarrow \infty} m(A_j)$$

*Proof.* Let  $A = \bigcap_{j=1}^{\infty} A_j$ .

**Claim** (Subclaim 1.).  $m(A) \leq \lim_{j \rightarrow \infty} m(A_j)$ .

*Proof.* Any COBC of  $A$  is a COBC of  $A_j$  for each  $j$ . For any  $x \geq j$ , any COBC of  $A_j$  is a COBC of  $A_x$ . For each  $j$ , we thus have

$$\begin{aligned} m(A) &= \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ is a COBC of } A \right\} \\ &\leq \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ is a COBC of } A_x, x \geq j \right\} \\ &= m(A_x) \text{ where } x \geq j \\ &\leq \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ is a COBC of } A_j \right\} \\ &= m(A_j). \end{aligned}$$

Since  $(m(A_j))_{j \in \mathbb{N}}$  is a monotonically decreasing sequence bounded below by  $m(A)$  (which is in turn bounded below by 0), we have  $m(A) \leq \lim_{j \rightarrow \infty} m(A_j)$   $\square$

**Claim** (Subclaim 2.).  $\lim_{j \rightarrow \infty} m(A_j) \leq m(A)$ .

*Proof.* Suppose  $\lim_{j \rightarrow \infty} m(A_j) > m(A)$ . Then  $\lim_{j \rightarrow \infty} m(A_j) = m(A) + \epsilon$  for some  $\epsilon > 0$ . Then for each  $j$  there exists some  $\epsilon_j > 0$  such that

$$\inf \left\{ \sum |B_{j,k}| \mid \{B_{j,k}\} \text{ is a COBC of } A_j \right\} = \inf \left\{ \sum |B_k| \mid \{B_k\} \text{ is a COBC of } A \right\} + \epsilon_j.$$

Since every COBC of  $A_j$  is a COBC of  $A$ , for each  $A_j$  there is a subset  $\alpha_j$  of  $A_j$  with nontrivial measure such disjoint from  $A$ .

Then the intersection  $\alpha = \bigcup_{j \in \mathbb{N}} \alpha_j$  is also disjoint from  $A$ . **I need to argue  $\alpha$  is nonempty with positive measure. But how?**

Then  $(\bigcap_{j \in J} A_j) \cap \alpha = \emptyset$ , but  $\alpha \subseteq A_j$  for every  $j$ , which is a contradiction.  $\square$

We have shown that  $m(A) \leq \lim_{j \rightarrow \infty} m(A_j) \leq m(A)$ , hence we have  $m(A) = \lim_{j \rightarrow \infty} m(A_j)$ .  $\square$

### Tao Exercise 7.2.4

**Claim.** *Let  $q > 1$  be a positive integer. Then the open and closed boxes  $(0, 1/q)^n$  and  $[0, 1/q]^n$  each have measure  $q^{-n}$ .*

### Tao Exercise 7.4.1

### Tao Exercise 7.4.2

### Tao Exercise 7.4.3

### Tao Exercise 7.4.4