Math 105. Lec 3. Measurable Set.

$A \subset \mathbb{R}^n$

Recall: · $m^*(A) := \inf \{ \Sigma_i |B_i|, \ \{B_i\}$ is countable collection of open boxes. covering $A \}$

- Properties of outer measure.

· $m^*(\phi) = 0$, $m^*(A) \geq 0$.

monotonicity.

· $A \subset B \Rightarrow m^*(A) \leq m^*(B)$

· countable sub-additivity. $A = \cup_{i=1}^{\infty} A_i$,

then $m^*(A) \leq \Sigma_{i=1}^{\infty} m^*(A_i)$.

· Def (measurable set): $E \subset \mathbb{R}^n$ is measurable iff. $\forall \, \underline{A} \subset \mathbb{R}^n$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Today:

Lemma 7.4.2 (half spaces in $\mathbb{R}^n$ are measurable).

i.e. $\{(x_1, \cdots, x_n) \mid x_n > 0\}$. is measurable in $\mathbb{R}^n$.

Pf for $(n=1)$: We want to show, $\forall A \subset \mathbb{R}$,

$$m^*(A) = m^*(A_+) + m^*(A_-)$$

where $A_+ = A \cap (0, \infty)$, $A_- = A \cap (-\infty, 0]$.

(1) first, $\because A = A_+ \sqcup A_-$, $\therefore$ by sub-additivity,

$$m^*(A) \leq m^*(A_+) + m^*(A_-).$$

(2) To show $m^*(A) \geq m^*(A_+) + m^*(A_-)$, it suffice to show,

$\forall \varepsilon > 0$,

$$m^*(A) + \varepsilon \geq m^*(A_+) + m^*(A_-)$$

Consider an open cover of A by $\{B_j\}$ by open boxes, such that $\sum_{j=1}^{\infty} |B_j| \leq m^*(A) + \varepsilon/2$.

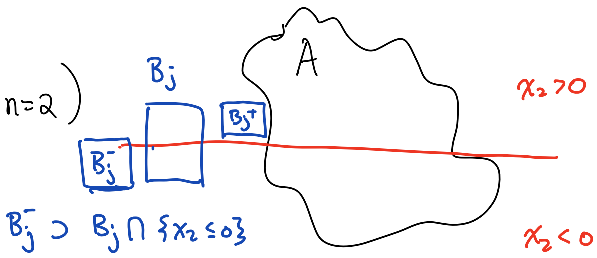
Define $B_j^+ = B_j \cap (0, \infty)$, $B_j^- = B_j \cap (-\infty, \frac{\varepsilon}{2^{j+1}})$.

then
- $B_j = B_j^- \cup B_j^+$, and $|B_j| + \frac{\varepsilon}{2^{j+1}} \geq |B_j^+| + |B_j^-| \geq |B_j|$
- $\cup B_j^+ \supset A_+$, $\cup B_j^- \supset A_-$

Thus. $m^*(A_+) + m^*(A_-) \leq \sum |B_j^+| + \sum |B_j^-| \leq \sum_{j=1}^{\infty} \left(|B_j| + \frac{\varepsilon}{2^{j+1}}\right)$

$$\leq \left(\sum |B_j|\right) + \frac{\varepsilon}{2} \leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = m^*(A) + \varepsilon. \quad \boxtimes$$

(Try $n=2$)



$A_+ = A \cap \{x_2 > 0\}$

$A_- = A \cap \{x_2 \leq 0\}$.

need to show $m^*(A) + \varepsilon \geq m^*(A_+) + m^*(A_-)$

one can do the same. ① get $\{B_j\}$, cover of A, with $m^*(A) + \frac{\varepsilon}{2} \geq \sum |B_j|$

② $B_j^+ = B_j \cap \{x_2 > 0\}$. $B_j^- = B_j \cap \{x_2 < \underline{\varepsilon_j}\}$

ε_j is chosen, such that

$$|B_j^-| + |B_j^+| \leq |B_j| + \frac{\varepsilon}{2^{j+1}}$$

the rest follows.

There is a better, more systematic approach.

Tao. Ex, 7.4.3: For A = open box in $\mathbb{R}^n$, prove

$$m^*(A) = m^*(A_+) + m^*(A_-).$$

should be easy, since $m^*(A) = |A|$, $m^*(A_+) = |A_+|$ ---, by direct computation

For general A, for any $\varepsilon > 0$, find $\{B_j\}$ cover of A.

s.t. $m^*(A) + \varepsilon \geq \sum |B_j| = \sum_j |B_j^+| + |B_j^-| = (\sum |B_j^+|) + (\sum |B_j^-|)$

define $B_j^+ = B_j \cap \{x_n > 0\}$, $B_j^- = B_j \cap \{x_n \leq 0\}$ (may not be open).

$\because A_+ \subset \bigcup B_j^+ \quad \therefore m^*(A_+) \leq \sum m^*(B_j^+) = \sum |B_j^+|$

$A_- \subset \bigcup B_j^- \quad \therefore m^*(A_-) \leq \sum m^*(B_j^-) = \sum |B_j^-|$.

$\therefore m^*(A_+) + m^*(A_-) \leq m^*(A) + \varepsilon$, ∎

<u>Lemma 7.4.4</u>: (property of measurable set).

(a) if $E \subset \mathbb{R}^n$ meas..., then E^c is meas.. ($E^c = \mathbb{R}^n \setminus E$)

(true, by definition, same tests for E & E^c).

(b) translation invariance. if E meas. then $\forall x \in \mathbb{R}^n$,

$x+E$ is measurable., $m(x+E) = m(E)$.

(Pf: outer measurable is translation invariant.

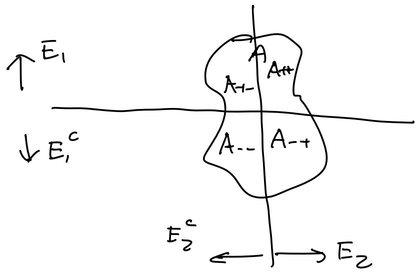
$\forall A \subset \mathbb{R}^n$, $m^*(A) \stackrel{?}{=} m^*(A \cap (x+E)) + m^*(A \cap (x+E)^c)$

$\Leftrightarrow m^*(A-x) = m^*((A-x) \cap E) + m^*((A-x) \cap E^c)$ this ✓. holds)

(c). If E_1 and E_2 are measurable, then $\underline{E_1 \cap E_2}$, $E_1 \cup E_2$ are meas.. ($E_1 \cup E_2 = (E_1^c \cap E_2^c)^c$)

(Pf: WTS $\forall A \subset \mathbb{R}^n$,

(*) $m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \setminus \underline{(E_1 \cap E_2)})$.



$A_{++} = A \cap E_1 \cap E_2, \qquad A_{+-} = A \cap E_1 \cap E_2^c$

$A_{--} = A \cap E_1^c \cap E_2, \qquad A_{--} = A \cap E_1^c \cap E_2^c.$

$A = A_{++} \sqcup A_{+-} \cup A_{-+} \cup A_{--}.$

$(*) \iff \underline{m^*(A)} = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})$

We can show.

using E_1 measurable $\quad m^*(A) = m^*(\overbrace{A_{+-} \cup A_{++}}^{A \cap E_1 =}) + m^*(\overbrace{A_{-+} \cup A_{--}}^{A \cap E_1^c =})$

$$\left.\begin{aligned} m^*(A_{+-} \cup A_{++}) &= m^*(A_{+-}) + m^*(A_{++}) \\ m^*(A_{-+} \cup A_{--}) &= m^*(A_{--}) + m^*(A_{-+}). \end{aligned}\right\} \because E_2 \text{ measurable.}$$

(still need more arguments $\because$).

<u>exercise</u> $\quad m^*(A_{+-} \cup A_{-+} \cup A_{--}) = m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$

(d). (Boolean algebra) finite intersection / union preserve measurability (✓ using induction on number of operand).

(e) every box (open or closed), is measurable.

$$\Big(\quad [a,b] = \underbrace{[a,\infty)}_{a+[0,\infty)} \cap \underbrace{(-\infty,b]}_{b+(-\infty,0]}. \quad \Big)$$

boxes are intersections of (translated) half spaces.

(f) if $m^*(E) = 0$, then E is measurable.

Pf: need to $\forall A \subset \mathbb{R}^n$, $\quad m^*(A) \underset{\le\vee}{=} m^*(A \cap E) + m^*(A \setminus E).$

only need $\quad m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E). \qquad (*)$

$\because m^*(A \cap E) \leq m^*(E) = 0. \quad \therefore m^*(A \cap E) = 0.$

$(*) \iff \quad m^*(A) \geq m^*(A \setminus E) \quad$ which is true by monotonicity.

Lemma 7.4.5 (finite additivity), if $E_1, \dots, E_n$ are disjoint measurable set. then $\forall A \subset \mathbb{R}^n$, $E = \bigcup_{i=1}^{n} E_i$

$$m^*(A \cap E) = \sum_{i=1}^{n} m^*(A \cap E_i)$$