

Q. 9.4. Suppose there exists N_0 such that $S_n \leq t_n$ for all $n \geq N_0$ (As n gets big, t_n gets bigger than S_n)

a) Prove that if $\lim S_n = +\infty$, then $\lim t_n = +\infty$

Given that $\lim S_n = +\infty$ we have
 $\lim S_n = +\infty$ i.e. for every $M > 0$, there
is a number N such that
 $n > N \rightarrow S_n > M$

~~lim~~ $\lim \left(\frac{1}{S_n}\right) = 0$ by 9.10
WTS: $\lim \left(\frac{1}{S_n}\right) = 0 \rightarrow \lim \left(\frac{1}{t_n}\right) = 0$

WTS: ~~$\exists M > 0$~~
Let $\epsilon > 0$ and let

WTS: for every $M > 0$ there exists a N
such that $n > N$ implies $t_n > M$.

Let N_0 exist such that $S_n \leq t_n$ $\forall n \geq N_0$

Further suppose $\lim S_n = +\infty$, then

For every $M > 0$ there exists a N such
that $n > N$ implies $S_n > M$.

Then for every $M > 0$ there exists

some $N' \geq N_0$ such that $t_n > M$

because ~~S_n~~ $t_n \geq S_n > M$ $\forall n \geq N_0$,

hence $\lim t_n = +\infty$ by the definition
of a limit being infinity. \square

$\exists N_0$ such that $s_n \leq t_n \forall n > N_0$

Q.9b) ^{if} Prove $\lim t_n = -\infty$ then $\lim s_n = -\infty$.

$\lim t_n = -\infty$, thus for each $M < 0$, there is an N such that $n > N \rightarrow t_n < M$

WTS: For each M , $\exists N'$ such that for $n > N'$, $s_n < M$

For each $M < 0$, there is some N such that $n > N \rightarrow t_n < M$

Let N_0 exist such that for $n > N_0$, $s_n \leq t_n$, then let N' exist such that $n > N' \rightarrow t_n < M$

~~$n > N_0$~~ $n > N' \rightarrow s_n \leq t_n$

$s_n \leq t_n < M \forall n > N' \geq N_0$

thus $s_n < M \forall n > N'$

and by the definition for negative infinity, limit being negative infinity,

$$\lim s_n = -\infty$$

□

Q.4c IF $\lim s_n$ and $\lim t_n$ exist, then
 $\lim s_n \leq \lim t_n$ WTS $\lim s_n - \lim t_n \leq 0$

Let $\lim s_n = s$ and $\lim t_n = t$

Then $\forall \epsilon > 0 \exists N_s$ such that $n > N_s$

implies $|s_n - s| < \epsilon$

and, ~~and~~ $\exists N_t$ such that $n > N_t$

implies $|t_n - t| < \epsilon$

We have some N_0 such that $s_n \leq t_n$

$\forall n > N_0$

Let ~~$s_n - t_n$~~ be the ^{sequence} ~~limit~~ defined by

$$s_n - t_n \quad \forall n > 0$$

WTS: ~~$\forall \epsilon > 0 \exists N$~~ $\forall \epsilon > 0 \exists N$ s.t. $\forall n > N \rightarrow$ $|s_n - t_n| < \epsilon$

Now: $|t_n - s_n| < \epsilon$

Let $\epsilon > 0$

Let $\epsilon > 0$ then there exists some $N' \geq N_0$

such that $s_n \leq t_n \quad \forall n \geq N'$

~~with~~ $0 \leq t_n - s_n \quad \forall n \geq N'$

~~$\lim(t_n - s_n) \neq$~~ $|t_n - s_n - (t - s)| < \epsilon$

a_n

$$\exists N \text{ such that } \forall n \geq N$$
$$\frac{a^{n+1}}{(n+1)!} \leq \frac{a^n}{n!} \quad \forall a \in \mathbb{R}$$

note: $\frac{a^n}{n!} > 0 \quad \forall n$

~~a^n~~

$$\frac{a^n \cdot a}{(n+1)n!} \leq \frac{a^n}{n!}$$

$$\frac{a}{n+1} \leq 1$$

For sufficiently large n , as n gets big

thus.

$$\frac{a^{n+1}}{(n+1)!} \leq \frac{a^n}{n!}$$

i.e. the sequence is monotonically decreasing and bounded below by zero, thus it converges

$$\text{thus } \lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0$$



167 Let S be a bounded nonempty subset of \mathbb{R} .
 Show that $\sup S$ is not in S . Prove
 there is a sequence (s_n) of points in
 S such that $\lim s_n = \sup S$.

S is bounded $\Rightarrow \sup S \in \mathbb{R}$

$\sup S - \frac{1}{n}$ is not an upper bound of
 S , (and is thus in S), so
 there is an element in S
 denoted s_n greater than $\sup S - \frac{1}{n}$.

We construct our sequence
 with this term, with $\sup S - \frac{1}{n}$
 as the n th term.

We have our sequence s_n
 with $\sup S > s_n \geq s_n = \sup S - \frac{1}{n}$,
 thus a sequence ~~exists with~~
 ~~s_n~~ exists with $\lim s_n = \sup S$

$$\frac{1}{n} < \frac{1}{n-1} + 1$$

$$\frac{1}{n} < \frac{1}{n-1} + 1$$

$$\frac{1}{n} < \frac{1}{n-1} + 1$$

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Q. Let S_n be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$. Prove σ_n is an increasing sequence.

$$\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

$$\sigma_{n+1} = \frac{1}{n+1}(s_1 + s_2 + \dots + s_n + s_{n+1})$$

$$\sigma_{n+1} - \sigma_n \geq 0$$

$$\frac{1}{n+1}(s_1 + s_2 + \dots + s_n + s_{n+1}) \geq \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

$$\frac{1}{n+1}(s_1 + s_2 + \dots + s_n) + \frac{s_{n+1}}{n+1} \geq \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

$$\frac{1}{n+1} + \frac{s_{n+1}}{(n+1)(s_1 + s_2 + \dots + s_n)} \geq \frac{1}{n}$$

$$\frac{1}{n+1} \left(1 + \frac{s_{n+1}}{s_1 + s_2 + \dots + s_n} \right) \geq \frac{1}{n}$$

$$1 + \frac{s_{n+1}}{s_1 + s_2 + \dots + s_n} \geq \frac{n+1}{n}$$

$$\frac{s_{n+1}}{s_1 + s_2 + \dots + s_n} \geq \frac{n+1}{n} - \frac{n}{n}$$

$$\frac{s_{n+1}}{s_1 + s_2 + \dots + s_n} \geq \frac{1}{n} \Rightarrow$$

$n s_{n+1} \geq s_1 + s_2 + \dots + s_n$
 because S_n is an increasing seq, s_n is greater than all s_i $\forall i \leq n$

thus n successive terms s_{n+1} is bigger than $\sum_{i=1}^n s_i$

thus σ_n is increasing

$$10.9 \quad s_1 = 1 \quad s_2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$$

$$s_3 = \frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{6} \quad s_4 = \frac{3}{4} \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{48}$$

b) We show s_n is monotonically decreasing
Proceed via induction

Our base cases are clearly true

$$s_1 > s_2 > s_3 > s_4$$

Assume this holds for $s_1 > s_2 > \dots > s_n$

Then

$$s_{n+1} = \frac{n}{n+1} s_n^2 = \frac{n}{n+1} \cdot s_n \cdot s_n$$

$s_n^2 \leq 1$ by our base cases

$\frac{n}{n+1} < 1$ clearly

therefore $s_{n+1} < s_n$ by induction.

We now show the sequence is bounded. Because $s_1 = 1$ and the sequence is decreasing, it is bounded above by 1 . Further, because the sequence consists of products of terms less than 1 and greater than 0 , the sequence is bounded below by zero.

10.9 C). By Thm 10.2 the sequence must converge, we evaluate the limit

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \lim_{n \rightarrow \infty} S_n = 0$$

$$\frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} = 1$$

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1) \quad \forall n \geq 1$

a) Find s_2, s_3, s_4

$$s_2 = \frac{1}{3}(1+1) = \frac{2}{3}$$

$$s_3 = \frac{1}{3}\left(\frac{2}{3}+1\right) = \frac{1}{3} \cdot \frac{5}{3} = \frac{5}{9}$$

$$s_4 = \frac{1}{3}\left(\frac{5}{9}+1\right) = \frac{1}{3} \cdot \frac{14}{9} = \frac{14}{27}$$

b) Show $s_n > \frac{1}{2}$ for all n via induction

Base case: We have shown $s_n > \frac{1}{2}$ for

s_1, s_2, s_3, s_4

Assume $s_n > \frac{1}{2}$ for all n

Then,

$$s_{n+1} = \frac{1}{3}(s_n + 1)$$

$$= \frac{1}{3}s_n + \frac{1}{3}$$

by our Inductive hypothesis $s_n > \frac{1}{2}$

$$\text{Thus } s_{n+1} > \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3}$$

$$> \frac{1}{6} + \frac{1}{3} = \frac{1}{6} + \frac{2}{6} = \frac{1}{2}$$

$$s_{n+1} > \frac{1}{2}$$

Thus, induction holds and $s_n > \frac{1}{2}$
for all $n \geq 1$

10.10 c) Show S_n is a decreasing sequence

$$s_1 > s_2 > s_3 > s_4$$

Claim: $S_n > S_{n+1} \quad \forall n \geq 1$

Proof via induction

$n=1, 2, 3$ have been shown true.

Assume $S_n > S_{n+1} \quad \forall n \geq 1$

Then,

$$S_{n+1} > S_{n+2}$$

$$\frac{1}{3}(S_{n+1}) > \frac{1}{3}(S_{n+1} + 1)$$

$$S_{n+1} > S_{n+1} + 1$$

$S_n > S_{n+1}$ true by our inductive hypothesis

Thus, induction holds

and $S_n > S_{n+1} \quad \forall n \geq 1$

and S_n is a decreasing sequence.

10.10 d) Show $\lim s_n$ exists and find it

c) has shown us that s_n is monotonically decreasing, further $s_1 = 1$, thus s_n is bounded above by 1, b) has shown that s_n is bounded below by $1/2$. Thus, s_n is bounded and monotonic, and $\lim s_n$ exists.

Thus $\lim_{n \rightarrow \infty} s_n = s$

$$\lim \left(\frac{1}{3}(s_n + 1) \right) = s$$

$$\lim \left(\frac{1}{3}(s + 1) \right) = s$$

$$\lim (s + 1) = 3s$$

$$\lim (1) = 2s$$

$$\lim (s) = \frac{1}{2}$$

$$\lim s_n = \frac{1}{2}$$

10.11 Let $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{4n^2}\right] t_n \quad \forall n \geq 1$

a) Show $\lim t_n$ exists

$$t_1 = 1, \quad t_2 = \frac{3}{4}, \quad t_3 = \frac{45}{64} \approx 0.703$$

$$t_4 = \frac{1575}{2304} \approx 0.683$$

Claim: t_n is decreasing, i.e. $t_n > t_{n+1} \quad \forall n \geq 1$

Proof via induction

Base Cases: $n = 1, 2, 3$ shown true

Assume $t_n > t_{n+1} \quad \forall n \geq 1$

$$t_{n+1} > t_{n+2}$$

$$\left[1 - \frac{1}{4(n+1)^2}\right] t_{n+1}$$

$$1 > 1 - \frac{1}{4(n+1)^2}$$

$$\frac{1}{4(n+1)^2} > 0 \quad \text{true } \forall n \geq 1$$

Thus induction holds and t_n is decreasing for all $n \geq 1$

Claim: $t_n > 0 \quad \forall n \geq 1$

Proof via induction:

Base Cases: $n = 1, 2, 3$, shown to hold

Assume $t_n > 0 \quad \forall n \geq 1$

$$t_{n+1} > 0$$

$$\left[1 - \frac{1}{4n^2}\right] t_n > 0$$

$$1 - \frac{1}{4n^2} > 0$$

$$4n^2 > 1 \quad \text{true } \forall n \geq 1$$

Thus induction holds and t_n is bounded below by zero. Observe that t_n is monotonically decreasing, further observe $t_1 = 1$, thus t_n is bounded above by 1, and thus $\lim t_n$ must exist.

10.11 b) $\lim t_n = ?$

Attempting to evaluate $\lim t_n$

results in $\lim t = \lim \left[1 - \frac{1}{4n^2}\right] \cdot \lim t$

$\lim t = \lim t$, which

is unhelpful, numerical analysis

shows it approaches, or is at

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2) Let a_n, b_n, c_n be three sequences such that $a_n \leq b_n \leq c_n$, and $L = \lim a_n = \lim c_n$. Show that $\lim b_n = L$.

From the definition of limits we have

$$\exists N \forall \epsilon > 0 \forall n > N \\ |a_n - L| < \epsilon \quad \text{and} \\ |c_n - L| < \epsilon$$

by expanding,

$$-\epsilon < a_n - L \leq c_n - L < \epsilon$$

because $a_n \leq b_n \leq c_n$ we have

$$-\epsilon < a_n - L \leq b_n - L \leq c_n - L < \epsilon$$

thus, $|b_n - L| < \epsilon$

and by the definition of the limit,

$$\lim b_n = L.$$