Math 104 Midterm 2 Apr 5, 2022

Name: ______________________________

1. (10 points each, 50 points total) True or False. If you think the following statement is true, give a proof; if you think it is false, give a counter-example.

   (a) Let (X, d) be any metric space, then every Cauchy sequence is convergent.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Let $(x_n), (y_n)$ be two sequences in $\mathbb{R}$, such that $f(x_n) = y_n$. If (y_n) converges, then (x_n) converges.

   (c) For any two sequences of points $x_n, y_n \in \mathbb{R}$, where x_n are distinct, there exists a continuous function $f(x_n) = y_n$.

   (d) Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of uniformly continuous functions. If $f_n(x)$ converges pointwise to $f(x)$, i.e., for any $x \in \mathbb{R}$, $\lim_n f_n(x) = f(x)$, then $f(x)$ is continuous.

   (e) If $A, B \subset \mathbb{R}$ are bounded subsets, such that for any $\epsilon > 0$, there exists $x \in A$ and $y \in B$ such that $|x - y| < \epsilon$, then $\bar{A} \cap \bar{B} \neq \emptyset$.

2. (15 points) Is the following sequence of functions uniformly convergent on $\mathbb{R}$?

$$f_n(x) = n \ln(1 + x^2/n)$$

   where ln is the natural log. You may use Taylor expansion that $\ln(1 + x) = x + \cdots$.

3. (15 points) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Prove that

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k f(\frac{k}{2n}) = 0$$

4. (20 points) Consider $\mathbb{R}^2$ be equipped with the following metric

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| & y_1 = y_2 \\ |x_1| + |y_1 - y_2| + |x_2| & y_1 \neq y_2 \end{cases}$$

   Draw the closed ball centered at $(1, 0)$ with radius 2. (10 points). Is it compact? Prove your statement.(10 points)

1. (1) False. $X = (0,1) \subset \mathbb{R}$ with induced metric.

the sequence $a_n = \frac{1}{n}$ is Cauchy, but not convergent.

(there is no element in X that serves as the limit).

(2) False. Let $f(x) = \frac{1}{1+x^2}$. $x_n = n$, $y_n = \frac{1}{1+n^2}$,

then $y_n \to 1$, but x_n does not converge.

(3) False. Let $x_n = \frac{1}{n}$, $y_n = n$.

If there exists a continuous functions $f(x)$, with

$f(x_n) = y_n$, then $f(\lim_n x_n) = \lim_n f(x_n) = \lim_n y_n = \lim_n y_n$

but the RHS does converge

(4) false. (don't confuse uniform convergence with uniform continuity). Say $f_n(x) = \frac{1}{1+nx^2}$. then

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0. \end{cases}$$

(5) True. $\bar{A}$ and $\bar{B}$ are bounded and closed subset of $\mathbb{R}$, hence compact. For $n = 1, 2, \cdots$, let $x_n \in A$, $y_n \in B$ be chosen, s.t. $|x_n - y_n| < \frac{1}{n}$. Pick a subsequence x_{n_k}, such that $\lim_k x_{n_k} = x$ exists. By definition of closure, $x \in \bar{A}$. $\lim_k y_{n_k} = \lim_k x_{n_k} + (y_{n_k} - x_{n_k}) = x + 0 = x$.

Hence, $x \in \bar{B}$. Thus, $\bar{A} \cap \bar{B} \supset \{x\} \neq \phi$.

2. Not uniformly continuous.

The pointwise limit exists, $\lim f_n(x) = x^2$. Indeed,

for each $x \in \mathbb{R}$, if $x = 0$, then $\lim\limits_{n} n \ln(1+\frac{x^2}{n}) = \lim n \cdot \ln 1 = 0$

if $x \neq 0$, then

$$\lim_{n} n \cdot \ln(1+\frac{x^2}{n}) = \lim_{n} n \cdot \left[\frac{x^2}{n} + \left(\frac{\ln(1+\frac{x^2}{n}) - \frac{x^2}{n}}{x^2/n} \right) \cdot \frac{x^2}{n} \right]$$

$$= x^2 + x^2 \lim_{n \to \infty} \frac{\ln(1+\frac{x^2}{n}) - \frac{x^2}{n}}{x^2/n} = x^2.$$

where the last step uses. $\lim\limits_{y \to 0} \frac{\ln(1+y) - y}{y} = 0$.

However, $\sup\limits_{x \in \mathbb{R}} |f_n(x) - f(x)| = \infty$, hence $f_n(x)$ doesn't converge to $f(x)$.

3. Since $f(x)$ on $[0,1]$ is continuous, and $[0,1]$ is compact, $f(x)$ is uniformly continuous. Thus, for any $\varepsilon > 0$, $\exists\ \delta$. s.t. if $|x_1 - x_2| < \delta$, $\overbrace{\text{we}}^{x_i \in [0,1]}$ have $|f(x_1) - f(x_2)| < \varepsilon$.

For n large enough, s.t. $\delta > \frac{1}{2n}$, we have

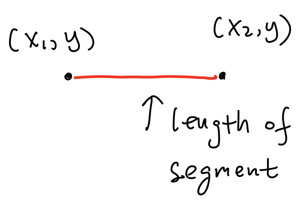
$$\left| \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k f\left(\frac{k}{2n}\right) \right| = \frac{1}{2n} \left| -f\left(\frac{1}{2n}\right) + f\left(\frac{2}{2n}\right) - \cdots \right|$$

$$\leq \frac{1}{2n} \Big[\underbrace{\left| -f\left(\tfrac{1}{2n}\right) + f\left(\tfrac{2}{2n}\right) \right| + \left| -f\left(\tfrac{3}{2n}\right) + f\left(\tfrac{4}{2n}\right) \right| + \cdots}_{n \text{ pairs}} \Big]$$

$$\leq \frac{1}{2n} \cdot n \cdot \varepsilon \quad \leq \frac{1}{2}\varepsilon.$$

Thus, $\lim_{n\to\infty} \left| \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k f\left(\frac{k}{2n}\right) \right| \leq \varepsilon/2$. Since $\varepsilon > 0$ is arbitrary, thus, $\lim_{n\to\infty} \left| \frac{1}{2n} \sum_{k=1}^{2n} (-1)^k f\left(\frac{k}{2n}\right) \right| = 0$

4. The metric measure distance in the following.



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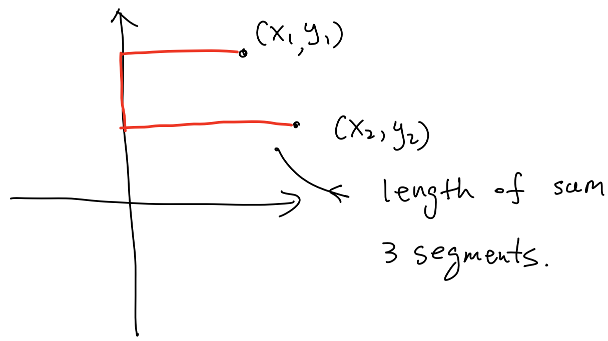
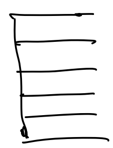
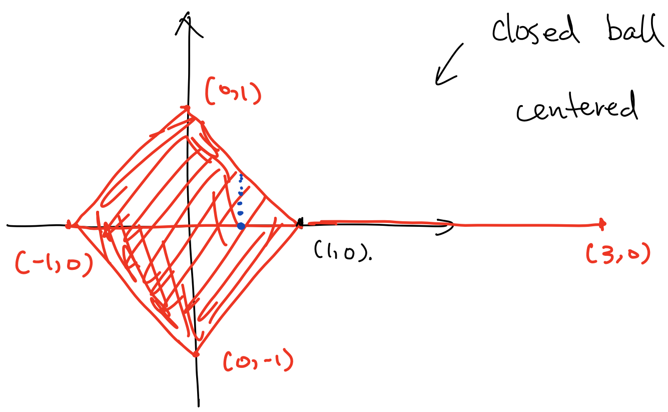


(Like walking on a comb,  , with ∞ many teeth).



Closed ball of radius 2 centered at $(1,0)$.

- Not compact. We can produce a sequence without convergent subsequence $(x_n, y_n) = \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{n}\right)$.

$$d\left((x_n, y_n), \ (x_m, y_m)\right) = |x_n| + |x_m| + |y_n - y_m|$$
$$= \frac{1}{2} + \frac{1}{2} + \left|\frac{1}{n} - \frac{1}{m}\right| \geq 1.$$

Hence there is no subsequence that is Cauchy, hence cannot be convergent.