

1. a) $F(x) = \cos(x) - x$ cont. on \mathbb{R} , take $[-\pi, \pi]$

$$F(0) = 1 - 0 = 1 \quad F\left(\frac{\pi}{2}\right) < 0 < F(\pi) \Rightarrow \text{By Bolzano 4.23, } \exists x \in [0, \frac{\pi}{2}] \text{ s.t. } F(x) = 0 \Rightarrow \cos x - x = 0 \Rightarrow \cos x = x$$

$$F\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

b) (a_n) convergent?

Bounded since $|\cos^n(x)| \leq 1$.

$\sum a_n$ diverge since for all a_n , $a_n > -1$, $\sum -1$ diverge with n odd $\Sigma = -1$; n even $\Sigma = 1$.

2. L(P, f) and U(P, f) is step function.

Hint 1: Suffice to show for constant

$$\lim_{n \rightarrow \infty} \int_a^b C \sin(nx) dx = \lim_{n \rightarrow \infty} \frac{C}{n} (\cos(ax) - \cos(bx)) = 0$$

Fix $\epsilon > 0$.

Then since $f \in \mathcal{R}$, $\exists P$ s.t. $0 \leq \int_a^b f dx - L(P, f) < \epsilon$. And $L(P, f) = \int_a^b S(x) dx$ for some step function. $S(x)$.

$$\left| \int_a^b (f(x) - S(x)) \sin nx dx \right| \leq \int_a^b (f(x) - S(x)) |\sin nx| dx \leq \int_a^b (f(x) - S(x)) dx < \epsilon$$

$$\Rightarrow -\epsilon + \int_a^b S(x) \sin nx dx \leq \int_a^b f(x) \sin nx dx \leq \epsilon + \int_a^b S(x) \sin nx dx$$

take \lim w.r. to $n \rightarrow \infty$, then $-\epsilon \leq \liminf \int_a^b f(x) \sin nx dx \leq \limsup \int_a^b f(x) \sin nx dx \leq \epsilon$. $\forall \epsilon > 0$

$$\text{Hence } \lim_{n \rightarrow \infty} \sup \int_a^b f(x) \sin nx dx = \lim_{n \rightarrow \infty} \inf \int_a^b f(x) \sin nx dx = \lim_{n \rightarrow \infty} \int_a^b f(x) \sin nx dx = 0$$

3. a) $F(x) = f(x)(g(x)h(x) - h(x)g(x)) - g(x)(f(x)h(x) - h(x)f(x)) + h(x)(f(x)g(x) - g(x)f(x))$

\hookrightarrow linear combination of f, g, h , hence F cont. on $[a, b]$, diff. on (a, b)

b) $F(a) = F(b)$, then by Rolle's Thm, $\exists x \in (a, b)$, s.t. $F'(x) = 0$.

c) Consider $h(x) = 1$ and apply part b to it.

4. a) Consider any Cauchy sequence in $(\mathcal{B}(X), d_B)$, $\{f_n\}$

By def of Cauchy, $\forall \epsilon > 0, \exists N > 0, \forall n, m > N, d_B(f_n, f_m) < \epsilon \Rightarrow \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon \Rightarrow |f_n(x) - f_m(x)| < \epsilon$ for all $x \in X$.

Hence $\{f_n\}$ is uniformly Cauchy, implies $\{f_n\}$ uniformly converge to some f .

And since $f_n \rightarrow f$ uniformly, then f should also be bounded hence in $\mathcal{B}(X)$.

b) By part a, we know all Cauchy hence convergent sequences in $\mathcal{B}(X)$ converge to some element in $\mathcal{B}(X)$.

Since $C(X) \subset \mathcal{B}(X)$, and by uniform convergence, if $\{f_n\} \subset C(X)$ and converge to f , $f \in C(X)$ since f is cont.

By Prop 11.7 (b), $C(X)$ is closed set in $\mathcal{B}(X)$.

c) Suppose A is a complete metric space, then any Cauchy sequence converges to some $a \in A$. Let B be a closed subset of A .

Then by definition of B , B contains limits of all convergent sequences in B , which is also Cauchy. Hence B contains limit for all Cauchy seq. in B .

Thus B is also a complete metric space.

Hence $C(X)$ is a complete metric space.

5. Suppose $[a, b]$ is of measure zero i.e. $\sum \text{vol}(U_k) < \epsilon$ for all $\epsilon > 0$. Then it should have a subcover of \mathcal{I} covering.

First note that by Heine-Borel in \mathbb{R} , $[a, b]$ is closed and bounded $\Rightarrow [a, b]$ is compact.

Suppose \exists a bad covering for $[a, b]$, $\{I_1, \dots\}$, then by compactness of $[a, b]$ in \mathbb{R} , then

$\{I_1, \dots, I_n\}$ is a finite bad covering for $[a, b]$. $\sum_{i=1}^n \text{length}(I_i) < (b-a)$.

Then take $I_1 \cup I_2 = J$ to replace I_1 and I_2 . And $\text{length}(J) \leq \text{length}(I_1) + \text{length}(I_2)$

By induction, there exists a single interval cover $[a, b]$, but its length is smaller than $b-a$, contradiction.

$\Rightarrow [a, b]$ is not of measure zero.

6. Consider $\epsilon = 1$. By unif. cont. of f , $|f(x) - f(y)| < \epsilon = 1$ if $|x - y| < \delta$.

Take $x \geq 1$, $\delta = \frac{x-1}{2} + 1$ means the δ of intervals with length $\frac{\delta}{2}$ in $[1, x]$. Then by that $|f(x) - f(1)| < \frac{x-1}{2} + 1$.

$$\Rightarrow |f(x) - f(1)| < \frac{x-1}{2} + 1$$

$$|f(x)| \leq f(1) + \frac{x-1}{2} = ax + b$$

Find some Mx such that for all $x \geq 1$, $Mx \geq ax + b$

$$\text{Then } |f(x)| \leq Mx \Rightarrow \frac{|f(x)|}{x} \leq M$$

7. a) Note that $e^{-nx} \cos nx \leq e^{-nx}$

So just consider e^{-nx} .

If $x \in (0, \infty)$, $0 \leq e^{-x} < 1$, hence $\sum_{n=1}^{\infty} \frac{1}{e^{nx}} = \frac{1}{e^x} + \frac{1}{e^{2x}} + \frac{1}{e^{3x}} + \dots$, by convergence of geometric series, $\sum_{n=1}^{\infty} e^{-nx}$ converges.

If $x = 0$, $e^{-nx} = 1$, $\sum_{n=1}^{\infty} 1 = \infty$, diverges.

If $x \in (-\infty, 0)$, $e^{-nx} = e^{ny}$ for $-x = y, y > 0$. Then e^{ny} is not bounded when $n \rightarrow \infty$, hence not convergent.

Thus $E = (0, \infty)$

b) No.

8.a) By Radic 4.29, f is monotonic increasing on $(0,1)$, $x \in (0,1)$, $f(x^-) = \sup_{0 < t < x} f(t)$, $f(x^+) = \inf_{x < t < 1} f(t)$.
 Since f is bounded on $(0,1)$ i.e. $t \in (0,1)$, $f(0) \leq f(t) \leq f(1)$, thus by completeness axiom, sup and inf exists, hence $f(x^-)$, $f(x^+)$ exists.

b) By a, $f(x^-)$ and $f(x^+)$ exist at all $x \in (0,1)$.

Then consider $r(x) \in \mathbb{Q}$ s.t. $f(x^-) < r(x) < f(x^+)$.

By 4.29, $f(x^+) \leq f(y^-)$ if $x < y$, then $r(x) < f(x^+) \leq f(y^-) < r(y)$. So $r(x) \neq f(x)$.

Hence a 1-1 correspondence with \mathbb{Q} , hence A is at most countable.

9.a) If $x \neq 0$, $f(x,y) = \frac{xy}{x^2+y^2}$ for any $y \in \mathbb{R}$.

xy is continuous, x^2+y^2 also continuous and nonzero, then $f(x,y)$ is continuous. (treat x as a constant).

If $x = 0$, $f(x,y) = \frac{0y}{0+y^2} = 0$ for $y \in \mathbb{R} \setminus \{0\}$.

Similarly $f(x,y)$ is cont.

Check continuity at $y = 0$.

$$\lim_{y \rightarrow 0} f(0,y) = \lim_{y \rightarrow 0} f(0,y) = \lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0.$$

L'Hospital

$$\lim_{y \rightarrow 0} \frac{0}{y^2} = \lim_{y \rightarrow 0} \frac{0}{2y} = \lim_{y \rightarrow 0} \frac{0}{2y} = 0.$$

$\lim_{y \rightarrow 0} f(x,y) = f(x,0)$, so $f(x,y)$ is also cont. at $y = 0$.

Same for fy .

b) Consider fixing $x = y$, then $f(x,y) = f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2}$, $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \frac{1}{2} \neq f(0,0)$.

Hence not continuous.

$$10. a_{n+1} = \sqrt{a_n^2 + \frac{1}{2^n}}$$

$$a_n^2 = a_{n-1}^2 + \frac{1}{2^{n-1}}$$

$$a_{n+1}^2 - a_n^2 = \frac{1}{2^n}$$

$$(a_{n+1} + a_n)(a_{n+1} - a_n) = \frac{1}{2^n}$$

Since a_n and a_{n+1} always ≥ 0 , and $\frac{1}{2^n} \geq 0$, hence $(a_{n+1} - a_n) \geq 0 \Rightarrow a_{n+1} \geq a_n$

Hence a_n is a monotone increasing sequence.

$$a_n = \sqrt{1 + \sum_{k=1}^{n-1} \frac{1}{2^k}}$$

$$a_1 = 1 \checkmark$$

$$\text{Ind. Step: } a_n = \sqrt{1 + \sum_{k=1}^{n-1} \frac{1}{2^k}}$$

$$a_{n+1} = \sqrt{1 + \sum_{k=1}^n \frac{1}{2^k} + \frac{1}{2^n}}$$

$$= \sqrt{1 + \sum_{k=1}^n \frac{1}{2^k}}$$

$$\lim_{n \rightarrow \infty} a_n = \sqrt{1 + \sum_{k=1}^{\infty} \frac{1}{2^k}} = \sqrt{2}.$$

11. All roots are in \mathbb{R} , then we rewrite $P(x) = c(x-r_1)\dots(x-r_n)$ since P only has deg P roots.

Assume $r_1 \leq r_2 \leq \dots \leq r_n$.

Hence $P(r_i) = P(r_i) = 0$, and P continuous on \mathbb{R} , by Bollo's Thm, exist $r'_i \in (r_i, r_{i+1})$ s.t. $P'(r'_i) = 0$.

Apply repetitively $n-1$ times, all $r'_i \in \mathbb{R}$, $i=1, \dots, n-1$.

12. Fix a $\epsilon > 0$ and $x_0 \in X$.

Claim: (f_n) converges to 0 pointwise.

$f_n(x) \geq 0 \forall n \in \mathbb{N}$ and $f_n(x) \geq f_{n+1}(x)$, (f_n) bounded for a fixed x_0 , and monotone decreasing. $\Rightarrow (f_n)$ converges.

By $\limsup_{n \rightarrow \infty} \{f_n(x) : x \in X\} = 0$. We know for any $\epsilon > 0$. $\exists N > 0$ s.t. $\sup \{f_n(x) : x \in X\} < \epsilon$ if $n > N$.

$\Rightarrow f_n(x) < \epsilon$ for all $x \in X$, $n > N$. Hence $f_n(x_0) < \epsilon$ for $n > N$. Hence by definition, $\lim_{n \rightarrow \infty} f_n(x_0) = 0$.

Consider this claim, together with $\limsup_{n \rightarrow \infty} \{f_n(x) : x \in X\} = \limsup_{n \rightarrow \infty} \{f_n(x) - 0 : x \in X\} = 0$, we conclude $(f_n) \rightarrow 0$ uniformly.

Then by alternating series test, $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$, $x \in X$ and $f_n(x) \geq 0$, $\lim_{n \rightarrow \infty} f_n(x) = 0$ for all $x \in X$.

Then $\sum_{n=0}^{\infty} (-1)^n f_n(x)$ converges uniformly.

13. f uniform cont. $\Rightarrow |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. Consider $\frac{1}{n} < \delta$, then $|f(\frac{k+1}{n}) - f(\frac{k}{n})| < \epsilon$ ($\frac{k+1}{n} - \frac{k}{n} = \frac{1}{n} < \delta$).

Hence for $n = 2m$ for some m :

$$|\frac{1}{n} \sum_{k=1}^n (-1)^k f(\frac{k}{n})| < \frac{1}{n} \cdot \frac{n}{2} \cdot \epsilon < \frac{\epsilon}{2} < \epsilon.$$

If $n = 2m+1$:

$$|\frac{1}{n} \sum_{k=1}^n (-1)^k f(\frac{k}{n})| < \frac{1}{n} \cdot \frac{n-1}{2} \cdot \epsilon + \frac{|f(1)|}{n} < \frac{\epsilon}{2} + \frac{|f(1)|}{n} \leftarrow \text{could find some } n \text{ such that } \frac{|f(1)|}{n} < \frac{\epsilon}{2}.$$

Therefore $|\frac{1}{n} \sum_{k=1}^n (-1)^k f(\frac{k}{n})| < \epsilon$ for $\epsilon > 0$, hence $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (-1)^k f(\frac{k}{n}) = 0$.

14. Consider $g(x) = f(x) - f(x + \frac{\pi}{2})$ is cont.

$$g(0) = f(0) - f(\frac{\pi}{2})$$

$$g(\frac{\pi}{2}) = f(\frac{\pi}{2}) - f(\pi) = f(\frac{\pi}{2}) - f(0) \text{ since } f(0) = f(0+\pi) = f(\pi)$$

If $g(0) = g(\frac{\pi}{2}) = 0$, then add T to both, i.e. $g(T)$ and $g(\frac{\pi}{2})$, to get 2 non-zero values.

WLOG assume $g(0)$ and $g(\frac{\pi}{2}) \neq 0$.

$g(0)$ and $g(\frac{\pi}{2})$ must have different signs, then by Rudin 4.23, $g(0) < 0 < g(\frac{\pi}{2})$ or $g(\frac{\pi}{2}) < 0 < g(0)$.

$\exists x_0 \in (0, \frac{\pi}{2})$ such that $g(x_0) = 0$ i.e. $f(x_0) = f(x_0 + \frac{\pi}{2})$.

15. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \dots + a_n \sin nx$

$$|f(x)| \leq |a_1 \sin x|$$

$$|f(x) - 0| \leq |\sin x - 0| \quad \text{Note: } f(0) = \sin 0 = 0$$

$$|f(x) - f(0)| \leq |\sin x - \sin 0|$$

$$\frac{|f(x) - f(0)|}{|x - 0|} \leq \frac{|\sin x - \sin 0|}{|x - 0|}$$

$$\lim_{x \rightarrow 0} \left| \frac{f(x) - f(0)}{x - 0} \right| \leq \lim_{x \rightarrow 0} \left| \frac{\sin x - \sin 0}{x - 0} \right|$$

$$\left| \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \right| \leq \left| \lim_{x \rightarrow 0} \frac{\sin x - \sin 0}{x - 0} \right| \quad \text{with diff. at } 0.$$

$$|f'(0)| \leq |\cos(0)|$$

$$|a_1 + 2a_2 + \dots + na_n| \leq 1.$$

16.

17. Consider $h(x) = f(x)e^{g(x)}$, which is also cont. on $[a, b]$ and diff. on (a, b) .

$$h(a) = f(a)e^{g(a)} = 0 = h(b) = f(b)e^{g(b)}$$

By Rolle's Thm, $\exists x \in (a, b)$ s.t. $h'(x) = 0$ i.e. $f'(x)e^{g(x)} + f(x)e^{g(x)} \cdot g'(x) = 0$

$\Rightarrow e^{g(x)} (f'(x) + f(x)g'(x)) = 0$ since $e^{g(x)} \neq 0$; hence $f'(x) + g'(x)f(x) = 0$ for some $x \in (a, b)$.

18. a) Suppose $f \in \mathcal{R}$