Numbers: 1.20

$\mathbb{N}$: natural numbers $\Rightarrow$ induction

$\mathbb{Z}$: integers

$\mathbb{Q}$: rationals: n/m, $n, m \in \mathbb{Z}$.

* denseness

prop If r is a rational number and is a root of

$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0 = 0$, $c_0, c_n \neq 0$, then

$d \mid c_n$, $c \mid c_0$

corollary If $r = \frac{c}{d} \neq 0$ is a root of monic polynomial, then r is an integer.

1.22

Def. maximum: $\forall \beta \in S, \alpha \geq \beta$ — may not exist ($\alpha \in S$)

minimum: $\forall \beta \in S, \alpha \leq \beta$

Def. upper bound: $\forall \beta \in S$, $\beta \leq \alpha$ — may not exist ($S = \mathbb{R}$)

lower bound: $\forall \beta \in S$, $\beta \geq \alpha$.

Def. $\sup(S) = \min \{\alpha \mid \alpha \text{ is an upper bound of } S\}$

$\inf(S) = \max \{\alpha \mid \alpha \text{ is a lower bound of } S\}$

Completeness Axiom: Every non-empty subset $S \subset \mathbb{R}$ that is bounded above has a $\sup(S)$.

corollary: If S bounded below, then $\inf(S)$ exists.

Archimedian Property If $a > 0$, $b > 0$, $a, b \in \mathbb{R}$, then for some $n \in \mathbb{N}$, $n \cdot a > b$

Sequence & Limits 1.27

Def. A sequence is: $a_1, a_2, a_3, \cdots, a_n \in \mathbb{R}, \forall n \in \mathbb{N}$

Def. A sequence (a_n) has limit $\alpha \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. for all $n > N$, $|a_n - \alpha| < \varepsilon$, denote by $\lim_{n\to\infty} a_n = \alpha$.

Def. (a_n) is a bounded sequence if $\exists M > 0$ s.t. $-M < a_n < M$ $\forall n \in \mathbb{N}$

Thm All convergent sequences are bounded.

Thm If $\lim_{n\to\infty} a_n = \alpha$, $k \in \mathbb{R}$, then $\lim_{n\to\infty} (k \cdot a_n) = k \cdot \alpha$.

Thm Let a_n, b_n be convergent sequences. $\lim a_n = \alpha$, $\lim b_n = \beta$.

① $\lim (a_n + b_n) = \alpha + \beta$

② $\lim (a_n \cdot b_n) = \alpha \cdot \beta$

③ If $b_n \neq 0$ for $\forall n$, $\beta \neq 0$, then $\lim \left(\frac{a_n}{b_n}\right) = \frac{\alpha}{\beta}$

1.29 Lemma If $b_n \neq 0$ $\forall n$, $\lim b_n \neq 0$, then $\frac{1}{b_n}$ is bounded.

Thm ① $\lim_{n\to\infty} \frac{1}{n^p} = 0$ $\forall p > 0$.

② $\lim a^n = 0$ if $|a| < 1$

③ $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

④ $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for $a > 0$.

Monotone and Cauchy

Def. (a_n) is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n, m > N$, $|a_n - a_m| < \varepsilon$

Def. increasing sequence: $a_{n+1} \geq a_n$

decreasing sequence: $a_{n+1} \leq a_n$

Thm All bounded monotone sequence are convergent.

Thm (a_n) Cauchy $\Longleftrightarrow$ (a_n) converges.

2.3 Limsup and liminf

Def. $\limsup a_n = \lim_{N\to\infty} (\sup_{n>N} \{a_n\})$, $\liminf a_n = \lim_{N\to\infty} (\inf_{n>N} \{a_n\})$

* $S_N = \sup_{n>N} \{a_n\}$ is decreasing (may not be bounded)

* if a_n bounded, then S_N bounded.

Rmk limsup exist in $\mathbb{R}$ or $-\infty$; liminf exists in $\mathbb{R}$ or $+\infty$

Lemma If (a_n) bounded, then limsup, liminf exist.

Lemma If (a_n) bounded, $\alpha^+ = \limsup a_n$, then $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$ we have $a_n < \alpha^+ + \varepsilon$

Similarly, $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n > N$, $a_n > \liminf a_n + \varepsilon$

Thm Let (a_n) be bounded. then $\lim a_n$ exists $\Longleftrightarrow$ $\limsup a_n = \liminf a_n$

2.5 Recursive sequence

show a_n monotone & bounded $\Rightarrow$ limit exists

then recursive equation take limit on both sides $\Rightarrow$ limit.

Subsequence

Def. subsequence: $t_k = s_{n_k}$, where $n_1 < n_2 < \cdots$. denote as $(s_{n_k})_k$

Lemma If (s_n) convergent, then any subsequence converges to same limit.

Lemma If $\alpha = \lim s_n$ exists, then $\exists$ a subsequence that is monotone.

Lemma For $\forall t \in \mathbb{R}$, (s_n) has a subsequence convergent to t

$\Longleftrightarrow$ $\forall \varepsilon > 0$, the set $\{n \mid |s_n - t| < \varepsilon\}$ is infinite.

Lemma For any sequence (s_n), there exists a monotone subsequence.

Thm (Bolzano - Weierstrass) 2.10

Every bounded sequence has a convergent subsequence.

Def: A subsequential limit is any real number or $\pm\infty$ that is the limit of a subsequence of (S_n).

Lemma $\exists$ a monotone subsequence whose limit is $\limsup S_n$.

Similarly, $\exists$ a monotone subsequence whose limit is $\liminf S_n$.

Thm (S_n) is a sequence. S = set of subsequential limits.

① S is non-empty.

② $\sup S = \limsup S_n$; $\inf S = \liminf S_n$.

③ $S = \{\alpha\} \Leftrightarrow \lim S_n$ exists and equals α.

Thm S is a closed set. For any sequence in S, (t_n) if $\lim t_n = t$ exists and $t \in \mathbb{R} \cup \{+\infty, -\infty\}$, then $t \in S$.

Limsup & Liminf (2) 2.12

In general: $\liminf S_n \leq \limsup S_n$.

If equal, then $\lim S_n$ exists.

Thm Let (S_n) has limit $s > 0$. Let (t_n) be a bounded sequence.

Then $\limsup (S_n \cdot t_n) = s \cdot \limsup (t_n)$

Thm Let (S_n) be a sequence of positive numbers. Then

$$\liminf \left(\frac{S_{n+1}}{S_n}\right) \leq \liminf (S_n)^{\frac{1}{n}} \leq \limsup (S_n)^{\frac{1}{n}} \leq \limsup \left(\frac{S_{n+1}}{S_n}\right)$$

Cheat Sheet.

Metric Space: (S,d), S is a set,

2.17

$d: S\times S \to \mathbb{R}_{\geq 0}$

① $d(x,y)>0$, $d(x,x)=0$

② $d(x,y)=d(y,x)$

③ $d(x,y)+d(y,z)\geq d(x,z)$

Lemma If (s_n) convergent to s, then s_n Cauchy

If every Cauchy seq. has a limit in $S \Rightarrow S$ complete

$\mathbb{R}^n$ is complete.

Lemma (s_n) be a seq in $\mathbb{R}^m$.

(1) (s_n) is Cauchy $\Leftrightarrow$ $(s_n^{(i)})$ is Cauchy in $\mathbb{R}$

(2) (s_n) convergent $\Leftrightarrow$ $(s_n^{(i)})$ is convergent.

Thm Every bounded seq. in $\mathbb{R}^m$ has a convergent subseq.

Topology: on a set S

Axioms: ① S,ϕ are open

② $\{U_i\}_{i\in I}$ is a collection of open subsets, $\bigcup_{i\in I} U_i$ open

③ $\{U_i\}_{i=1}^N$ is a finite collection of open subsets, $\bigcap_{i=1}^N U_i$ open.

Topology for metric space: (S,d) metric space, $\forall r>0, p\in S$

$B_r(p)=\{x\in S \mid d(p,x)<r\}$ open.

satisfy

* $U\subset S$ is open if $\forall p\in U, \exists r>0$ s.t. $B_r(p)\subset U$

2.24

Def. $p\in E$ is an interior point of E if $\exists \delta>0$ s.t. $B_\delta(p)\subset E$

E° = set of interior points of E.

Def. $E\subset S$ is open if $E=E^\circ$

Def. $E\subset S$ is closed if $E^c=S\setminus E$ is open.

① S,ϕ are closed.

② $\bigcap_\alpha F_\alpha$ closed if $\{F_\alpha\}$ closed.

③ $\bigcup_{i=1}^N F_i$ closed if $\{F_i\}_{i=1}^N$ closed.

Def. $E\subset S$, $p\in S$ is a limit point of $E \Leftrightarrow \forall\delta>0, B_\delta^\times(p)\cap E\neq\phi$

i.e. $\exists q\in E, q\neq p, d(p,q)<\delta$.

E' = set of limit points.

Def. the closure of E is the intersection of all closed subsets containing E.

$\overline{E}/E^- = \bigcap_{\substack{F\subset S\\ F\supseteq E}} F$ where F closed.

Property: $\overline{E}=E\cup E'$

Def. boundary of E: $\partial E=\overline{E}\setminus E^\circ$

Properties: ① E closed $\Leftrightarrow E=\overline{E}$

② E closed $\Leftrightarrow$ $\forall$ convergent seq. $x_n\to x$ in S. if $x_n\in E\ \forall n$, then $x\in E$

Compact Set

Def. $\{G_\alpha\}$ a collection of open sets. It's an open cover of E if $E\subset\bigcup_\alpha G_\alpha$

Def. $K\subset S$ is compact if $\forall$ open cover of K, $\exists$ a finite subcover. i.e. $K\subset G_{\alpha_1}\cup G_{\alpha_2}\cup\cdots\cup G_{\alpha_n}$ ($\{G_\alpha\}$ open cover)

eg. $[0,1]$ compact, $(0,1]$ not compact

Def. $E\subset S$ is sequentially compact if any seq. in E has a convergent subseq. in E

Thm E compact $\Leftrightarrow$ E seq. compact

Thm $E\subset\mathbb{R}^n$ is compact $\Leftrightarrow$ E closed and bounded

Lemma For $\forall$ open cover $\{G_\alpha\}$ of E, $\exists\delta>0$ s.t. $\forall x\in E$, $B_\delta(x)$ is contained in some G_α.

Lemma $\forall\delta>0$, $\exists$ finitely many points $x_1,\cdots,x_N$ s.t. $E\subset B_\delta(x_1)\cup\cdots\cup B_\delta(x_N)$

2.5

Remark "open"/"closed" depend on ambient space "compact" doesn't.

Def. X is connected if $\forall S\subset X$, if S is both open and closed, then $S=X$ or ϕ.

eg. $X=[0,1]\cup[2,3]$ not connected.

$U\cap V=\phi \Rightarrow U\cup V=U\amalg V$

X is connected $\Leftrightarrow$ If $X=U\amalg V$ and U,V are open, then one of U,V is ϕ.

3.12

Induced Topology: $S\subset X$, $U\subset S$ is open in $S\Leftrightarrow\exists\tilde{U}\subset X$ open in X s.t. $U=\tilde{U}\cap S$

Corollary (1) If $S\subset X$ open in X, $U\subset S$ open in $S\Leftrightarrow U$ open in X

(2) If $S\subset X$ closed in X, $E\subset S$ closed in $S\Leftrightarrow E$ closed in X.

Def. $E\subset X$ with induced topology. E is a connected subset $\Leftrightarrow$ E as a topological space is connected.

Lemma E connected $\Leftrightarrow$ E can't be written as $A\cup B$ where $\overline{A}\cap B=\phi$ and $A\cap\overline{B}=\phi$

Thm $E\subset\mathbb{R}$ is connected $\Leftrightarrow$ $\forall x,y\in E, x<y, [x,y]\subset E$

Series [2.16]

Def. Series is an infinite sum $\sum_{n=1}^{\infty} a_n$

partial sum: $s_n = \sum_{i=1}^{n} a_i$

Def A series converge to α if the partial sum converges.

Cauchy $\Leftrightarrow \forall \varepsilon>0, \exists N>0$ s.t. $\forall n,m>N$, $|\sum_{i=n+1}^{m} a_i| < \varepsilon$

Def If $\sum |a_n| < \infty$, then $\sum a_n$ converges absolutely.

1. Comparison test

① $\sum a_n < \infty$, $a_n > 0$, $b_n \in \mathbb{R}$, $|b_n| \leq a_n$

$\Rightarrow \sum b_n < \infty$

② $\sum a_n = \infty$, $a_n > 0$, $b_n \geq a_n$

$\Rightarrow \sum b_n = \infty$

2. Ratio test.

① If $\limsup |\frac{a_{n+1}}{a_n}| < 1$, then $\sum a_n$ converges.

② If $\liminf (|\frac{a_{n+1}}{a_n}| > 1)$, then $\sum a_n$ diverges.

Otherwise, No info.

3. Root test.

Let $\alpha = \limsup |a_n|^{\frac{1}{n}}$

Then $\sum a_n$ (1) converges absolutely if $\alpha < 1$

(2) diverges if $\alpha > 1$

(3) No info if $\alpha = 1$

Lemma If $\sum a_n$ converges, then $|a_n| \to 0$. (converse X)

Alternating Series test:

Thm (a_n) monotone decreasing, Assume $\lim a_n = 0$. Then,

$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 \cdots$ converges.

Thm $\sum \frac{1}{n^p} < \infty$ if $p > 1$

Uniform continuous. [3.10]

Def. $f: X \to Y$. uniform continuous if $\forall \varepsilon>0, \exists \delta>0$ s.t.

$\forall p,q \in X$ with $d_X(p,q) < \delta$, $d_Y(f(p), f(q)) < \varepsilon$.

Thm $f: X \to Y$ continuous and X compact, then f uniform continuous.

Prop If $f: X \to Y$ uniform continuous, $S \subset X$, then $f|_S : X \to Y$ is uniform continuous.

Thm If $f: X \to Y$ continuous, if $E \subset X$ connected, $f(E)$ connected

[3.3] Continuous Functions

Def. $(X, d_X), (Y, d_Y)$ metric spaces. $E \subset X$. $f: E \to Y$

limit of function: Suppose $p \in E'$, $f(x) \to q \in Y$ as $x \to p$ if

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in E$, $0 < d_X(x,p) < \delta \Rightarrow d_Y(f(x), q) < \varepsilon$.

Write: $\lim_{x \to p} f(x) = q$.

Thm: Following are equivalent:

(1) $\lim_{x \to p} f(x) = q$

(2) $\forall$ seq. $\in E$ convergent to p (p_n), $\lim(p_n) = p$, we have $\lim_{n\to\infty} f(p_n) = q$

Corollary: If f has limit point at p, it's unique.

Thm Suppose $f, g: E \to \mathbb{R}$, $p \in E'$, $\lim_{x\to p} f(x) = A$, $\lim_{x \to p} g(x) = B$

(1) $\lim_{x\to p} f(x) + g(x) = A + B$

(2) $\lim_{x \to p} f(x) g(x) = AB$.

(3) $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{A}{B}$ if $g(x) \neq 0$ $\forall x \in E$ and $B \neq 0$.

(4) $\forall c \in \mathbb{R}$, $\lim_{x \to p} c f(x) = cA$

Continuity:

Def 1: $f: E \to Y$. $p \in E$, $f(p) = q$. $(X, d_X)(Y, d_Y)$ metric spaces.

f is continuous at p if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in E$ with $d_X(x,p) < \delta$

$\Rightarrow d_Y(f(x), q) < \varepsilon$

Def 2: If $p \in E$ is a limit point of E

f is continuous at $p \Leftrightarrow \lim_{x \to p} f(x) = f(p)$

Def 3: f is continuous $\Leftrightarrow$ for $\forall$ open set $V \subset Y$, $f^{-1}(V)$ is open in X

(f continuous on E if f continuous at every point in E)

Lemma If $f: A \to B$, $E \subset A$, $F \subset B$, then $f(E) \subset F \Leftrightarrow E \subset f^{-1}(F)$

Thm. $f: X \to Y$, $g: Y \to Z$ continuous. $h: X \to Z$ - $h(x) = g(f(x))$

Then h is also continuous.

Thm $f, g: X \to \mathbb{R}$ continuous, then $f+g$, $f-g$, fg are continuous.

If $g(x) \neq 0$ for $\forall x \in X$, then $\frac{f}{g}$ continuous.

Thm $f: X \to \mathbb{R}^n$, with $f(x) = (f_1(x), \cdots, f_n(x))$.

Then f continuous $\Leftrightarrow f_i$ continuous for $\forall i$

[3.5] **Thm** $f: X \to Y$ continuous. If $E \subset X$ compact, then $f(E) \subset Y$ compact.

Corollary If $f: X \to \mathbb{R}$ continuous, X compact, then $\exists p, q \in X$ s.t.

$f(p) = \max(f(x))$, $f(q) = \min(f(x))$

Remark If $K \subset \mathbb{R}$, K compact, then $\sup(K) \in K$, $\inf(K) \in K$

*continuous f send compact in X to compact in Y. But given $E \subset Y$ compact, $f^{-1}(E)$ not guaranteed to be compact.

f continuous $\Leftrightarrow f^{-1}(\text{open})$ is open

$f^{-1}(\text{closed})$ is closed.

Thm If $f: [a,b] \to \mathbb{R}$, continuous. $f(a) < f(b)$

$f(a) < c < f(b)$ then $\exists x \in (a,b)$ s.t. $f(x) = c$

Def. $f: (a,b) \to \mathbb{R}$

(a) $\forall x \in [a,b)$, $f(x+) = q$ if $\forall$ seq. (t_n) in (x,b) that converge to x, $\lim_n f(t_n) = q$

(b) $\forall x \in (a,b]$, $f(x-) = q$ if $\forall$ seq. (t_n) in (a,x), that converge to x, $\lim_n f(t_n) = q$

Def. $f: (a,b) \to \mathbb{R}$, $x \in (a,b)$ and f not continuous at x_0.

f has 1st kind discontinuity if both $f(x_0+)$, $f(x_0-)$ exist.

otherwise, f has 2nd kind discontinuity.

3.17

Def. $f: \mathbb{R} \to \mathbb{R}$ monotone increasing if $\forall x > y$, $f(x) \geq f(y)$

decreasing if $\forall x > y$, $f(x) \leq f(y)$

Thm Suppose $f: (a,b) \to \mathbb{R}$ monotone increasing, then $\forall x \in (a,b)$, $f(x-)$, $f(x+)$ exist.

(1) $f(x-) = \sup\{f(t) \mid t < x\}$

(2) $f(x+) = \inf\{f(t) \mid t > x\}$

$f(x-) \leq f(x+)$

(3) Given $x < y$ in (a,b), $f(x+) \leq f(y-)$

Corollary: $f(x)$ only has 1st kind discontinuity if f monotone.

Prop. If f monotone, at most countably many discontinuities.

Seq. and Convergence of Function 3.15

Def. A seq. of functions $f_n \in Map(\mathbb{R}, \mathbb{R})$, converges to f pointwise if $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$ i.e. $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$

f_n converges to f uniformly if $\lim_{n\to\infty} d_\infty(f_n, f) = 0$

i.e. if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n > N$, $\forall x \in X$

$$|f_n(x) - f(x)| < \varepsilon$$

Thm Cauchy $\Leftrightarrow$ Convergence.

Suppose $f_n: X \to Y$, $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n, m > N$, $\forall x \in X$, $|f_n(x) - f_m(x)| < \varepsilon$

Then f_n converges uniformly.

Thm Suppose $f_n \to f$ pointwise, then

$f_n \to f$ uniformly $\Leftrightarrow \lim_{n\to\infty} (\sup_x |f_n(x) - f(x)|) = 0$

Thm Weierstrass M-test.

Suppose $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $\forall x \in X$

If $\exists M_n > 0$ s.t. $\sup_x |f_n(x)| \leq M_n$, $\sum_n M_n < \infty$

then $F_N(x) = \sum_{n=1}^{N} f_n(x)$ converges to $f(x)$ uniformly.

Thm Suppose $f_n \to f$ uniformly on E in X. x a limit point of E.

Suppose $\lim_{t\to x} f_n(t) = A_n$, then

① $\{A_n\}$ converges. let $A = \lim_n A_n$

② $\lim_{t\to x} f(t) = A$

i.e. $\lim_{t\to x} \underbrace{\lim_{n\to\infty} f_n(t)}_{f(t)} = \lim_{n\to\infty} \underbrace{\lim_{t\to x} f_n(t)}_{A_n}$

Thm Suppose $\{f_n\}$ is a seq. of continuous functions on E and $f_n \to f$ uniformly, then f continuous.

Thm Suppose K compact and

(a) $\{f_n\}$ is a seq. of continuous function on K

(b) $\{f_n\}$ converges pointwise to a continuous function f on K

(c) $f_n(x) \geq f_{n+1}(x)$ $\forall x \in K$, $\forall n = 1, 2, \cdots$

Then $f_n \to f$ uniformly on K.

Derivatives 4.7

Def. Let $f: [a,b] \to \mathbb{R}$ be a real-valued function.

The derivative is $\forall x \in [a,b]$, $f'(x) = \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$

If $f'(x)$ exists, f is differentiable at x.

Prop If $f: [a,b] \to \mathbb{R}$ differentiable at $x_0 \in [a,b]$, then f is continuous at x_0. i.e. $\lim_{x\to x_0} f(x) = f(x_0)$

Thm. Let $f, g: [a,b] \to \mathbb{R}$ differentiable at $x_0 \in [a,b]$. Then.

① $\forall c \in \mathbb{R}$, $(c \cdot f)'(x_0) = c \cdot f'(x_0)$

② $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

③ $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

④ If $g(x_0) \neq 0$, then $\left(\frac{f}{g}\right)'(x_0) = \frac{(f'g - fg')(x_0)}{g^2(x_0)}$

Rmk $(fg)' = f'g + fg'$

Thm Suppose $f: [a,b] \to \mathbb{R}$, $g: I \to \mathbb{R}$, $I \subset \mathbb{R}$, $x_0 \in [a,b]$, $f(x_0) = y_0$, $f([a,b]) \subset I$, $f'(x_0)$, $g'(y_0)$ exist. Then $h = g \circ f: [a,b] \to \mathbb{R}$ is differentiable at x_0. $h'(x_0) = g'(y_0) \cdot f'(x_0)$

Mean Value Theorem

Def. $f: [a,b] \to \mathbb{R}$. f has a local maximum at $p \in [a,b]$ if $\exists \delta > 0$ and $\forall x \in [a,b] \cap B_\delta(p)$, $f(x) \le f(p)$.

prop $f: [a,b] \to \mathbb{R}$. If f has a local max at $p \in (a,b)$, and $f'(p)$ exists, then $f'(p) = 0$.

* endpoints may have $f'(p) \ne 0$.

Thm (Rolle) $f: [a,b] \to \mathbb{R}$ continuous and differentiable in (a,b). If $f(a) = f(b)$, then $\exists c \in (a,b)$ s.t. $f'(c) = 0$.

4.9

Thm (MVT): Let $f, g: [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b). Then $\exists c \in (a,b)$ s.t.

$$[f(a) - f(b)] \cdot g'(c) = [g(a) - g(b)] \cdot f'(c)$$

Thm Let $f: [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b). Then $\exists c \in (a,b)$ s.t. $[f(b) - f(a)] = (b-a) f'(c)$

Corollary $f: [a,b] \to \mathbb{R}$ continuous, f' exists for (a,b), $|f'(x)| \le M$ for some M. Then f is uniformly continuous.

Corollary $f: [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b).
If $f'(x) \ge 0$ $\forall x \in (a,b)$, then f is monotone increasing.
If $f'(x) > 0$ $\forall x \in (a,b)$, then f is strictly increasing.

Intermediate Value Theorem

Thm f differentiable over $[a,b]$ with $f'(a) < f'(b)$. Then, for each $\mu \in (f'(a), f'(b))$, $\exists c \in (a,b)$ s.t. $f'(c) = \mu$.

L'Hospital's Rule

Thm Let $f, g: (a,b) \to \mathbb{R}$ differentiable, $g'(x) \ne 0$ on (a,b).
If either is true: (1) $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$

(2) $\lim_{x \to a} g(x) = +\infty$

And if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \in \mathbb{R} \cup \{+\infty, -\infty\}$

then $\lim_{x \to a} \frac{f(x)}{g(x)} = A$

Higher Derivatives 4.14

Def. If $f'(x)$ differentiable at x_0, then $f''(x_0) = (f')'(x_0)$

Similarly, if the $(n-1)^{th}$ derivative differentiable at x_0, $f^{(n)}(x_0) = (f^{(n-1)})'(x_0)$.

Def. f is a smooth function on (a,b) if $\forall x \in (a,b)$, $\forall k \in \{1, 2, \cdots\}$ $f^{(k)}(x)$ exists.

Taylor Theorem Let $f: [a,b] \to \mathbb{R}$, assume $f^{(n-1)}(x)$ exists and continuous $\forall x \in [a,b]$. $f^{(n)}(x)$ exists $\forall x \in [a,b]$

Then, $\forall \alpha \in [a,b]$, define $P_{\alpha,n}(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$

Then, $\forall \beta \in [a,b]$, if $\beta \ne \alpha$, $\exists \gamma$ between α and β s.t.

$$f(\beta) - P_{\alpha,n-1}(\beta) = \frac{f^{(n)}(\gamma)}{n!} (\beta - \alpha)^n$$

Taylor Series for smooth function

If f is a smooth function on (a,b) and $\alpha \in (a,b)$. We can form:

$$P_\alpha(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k$$

* ① RHS not guaranteed to converge
② If RHS converges, it may not equal to $f(x)$

4.16

Prop Consider power series $\sum_n c_n \cdot z^n$. Let $\alpha = \limsup_{n \to \infty} |c_n|^{\frac{1}{n}}$. $R = \frac{1}{\alpha}$

Then the series is convergent if $|z| < R$
divergent if $|z| > R$ (if $= R$ no info)

Weierstrass Approximation Thm If f continuous on $[a,b]$, then $\exists$ sequence of polynomial $f_n(x)$ s.t. $f_n \to f$ uniformly on $[a,b]$.

Riemann Integral

Def. Let $[a,b] \subset \mathbb{R}$ be a closed interval. A partition P of $[a,b]$ is a finite set of numbers in $[a,b]$: $a = x_0 \le x_1 \le x_2 \le \cdots \le x_n = b$

Define $\Delta x_i = x_i - x_{i-1}$, $i = 1, 2, \cdots, n$

Consider f real and bounded

Def Given $f: [a,b] \to \mathbb{R}$ bounded and partition $P = \{x_0 \le x_1 \le \cdots \le x_n\}$

Define $U(P,f) = \sum_{i=1}^n \Delta x_i \cdot M_i$, $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$

$L(P,f) = \sum_{i=1}^n \Delta x_i \cdot m_i$, $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

Def. $U(f) = \overline{\int_a^b} f dx = \inf U(P,f)$

$L(f) = \underline{\int_a^b} f dx = \sup L(P,f)$

A function f is Riemann integrable if $U(f) = L(f)$.

Stieltjes Integral 4.21

Def. Let $\alpha: [a,b] \to \mathbb{R}$ be an increasing function. $f: [a,b] \to \mathbb{R}$ bounded. P partition.

$U(P, f, \alpha) = \sum_{i=1}^n M_i \cdot \Delta\alpha_i$ where $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

$L(P, f, \alpha) = \sum_{i=1}^n m_i \cdot \Delta\alpha_i$

$U(f,\alpha)$. $L(f,\alpha)$ defined similarly as before.

Def. If $U(f,\alpha)=L(f,\alpha)$, f is integrable w.r.to α.

Write $f\in R(\alpha)$ on $[a,b]$

Rmk If $\forall x\in[a,b]$, $m\le f(x)\le M$, then

$$m(\alpha(b)-\alpha(a)) \le \sum_{i=1}^n m_i\cdot\Delta\alpha_i \le \sum_{i=1}^n M_i\,\Delta\alpha_i \le M(\alpha(b)-\alpha(a))$$

Def. Let P,Q be 2 partitions of $[a,b]$

$P=\{a=x_0<x_1<\cdots<x_n=b\}$

$Q=\{a=y_0<y_1<\cdots<y_m=b\}$

We say Q is a refinement of P if $P\subset Q$ as subsets of $[a,b]$

Given 2 partitions P_1,P_2, let $P_1\cup P_2$ denote their common refinement.

Lemma If Q is a refinement of P, then

$L(P,f,\alpha)\le L(Q,f,\alpha)$

$U(P,f,\alpha)\ge U(Q,f,\alpha)$

Thm $L(f,\alpha)\le U(f,\alpha)$

Thm (Cauchy condition) $f\in R(\alpha)\Leftrightarrow \forall\varepsilon>0\ \exists$ partition P s.t.

$$U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon.$$

Lemma (1) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$, then $\forall$ refinement Q of P, we have $U(Q,f,\alpha)-L(Q,f,\alpha)<\varepsilon$.

(2) If $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$, and let $s_i,t_i\in[x_{i-1},x_i]$, $\forall i=1,2,\cdots,n$, then

$$\sum_{i=1}^n|f(s_i)-f(t_i)|\cdot\Delta\alpha_i<\varepsilon.$$

(3) If $f\in R(\alpha)$, $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$, $s_i\in[x_{i-1},x_i]$ $\forall i$, then

$$\left|\sum f(s_i)\cdot\Delta\alpha_i-\int f\,d\alpha\right|<\varepsilon.$$

Thm If f continuous on $[a,b]$, then $f\in R(\alpha)$ on $[a,b]$.

Thm If f is monotonic on $[a,b]$, and α is continuous, then $f\in R(\alpha)$

4.33

Thm If f is discontinuous at finitely many points, and α is continuous where f is discontinuous, then $f\in R(\alpha)$

Thm $f:[a,b]\to[m,M]$ and $\phi:[m,M]\to\mathbb{R}$ continuous.

If $f\in R(\alpha)$, then $h=\phi\circ f\in R(\alpha)$

property: $\int f\,d\alpha$ is linear in both f and α.

Thm (1) If $f_1,f_2\in R(\alpha)$, and $c\in\mathbb{R}$, then

$f_1+f_2\in R(\alpha)$, $\int f_1+f_2\,d\alpha=\int f_1\,d\alpha+\int f_2\,d\alpha$

$c\cdot f_1\in R(\alpha)$, $\int c\cdot f_1\,d\alpha=c\cdot\int f_1\,d\alpha$

(2) Linearity in α similar.

(3) If $f,g\in R(\alpha)$ and $f(x)\le g(x)\ \forall x\in[a,b]$, then $\int f\,d\alpha\le\int g\,d\alpha$

Thm (1) If $f,g\in R(\alpha)$, then $fg\in R(\alpha)$

(2) If $f\in R(\alpha)$, then $|f|\in R(\alpha)$ and $|\int f\,d\alpha|\le\int|f|\,d\alpha$

Thm If $f:[a,b]\to\mathbb{R}$ and continuous at $s\in(a,b)$, and $\alpha(x)=I(x-s)$, then $\int f\,d\alpha=f(s)$

$I(x)=\begin{cases}0 & x\le0\\ 1 & x>0.\end{cases}$

Thm Let $c_n\ge0$ for $n=1,2,\cdots$. $\sum c_n<\infty$. Let (s_n) be a sequence of distinct points in $[a,b]$.

$$\alpha(x)=\sum_{n=1}^\infty c_n\cdot I(x-s_n)=\sum_{n:x>s_n}c_n.$$

Let $f:[a,b]\to\mathbb{R}$ be continuous, then

$$\int f\,d\alpha=\sum_{n=1}^\infty c_n\cdot f(s_n)$$

4.38

Thm Let α increasing. $\alpha'\in R$ (Rienam Integrable) on $[a,b]$. f is a bounded real function on $[a,b]$. Then

$$f\in R(\alpha)\Leftrightarrow f\cdot\alpha'\in R$$

and in this case $\int_a^b f\,d\alpha=\int_a^b f\cdot\alpha'\,dx$

Thm (change of variable) Let α increasing on $[a,b]$, $f\in R(\alpha)$. Suppose $\varphi:[A,B]\to[a,b]$ is a strictly increasing continuous function (i.e. bijection)

Define $\beta(y)=\alpha(\varphi(y))$, $g(y)=f(\varphi(y))$

Then $g\in R(\beta)$ and $\int_A^B g\,d\beta=\int_a^b f\,d\alpha$.

Relation between Differentiation and Integration

Thm 6.20 Let $f\in R$ on $[a,b]$. For $a\le x\le b$, let $F(x)=\int_a^b f(t)dt$.

Then: ① $F(x)$ is continuous on $[a,b]$

② If $f(x)$ continuous at $x_0\in[a,b]$, then $F(x)$ is differentiable at x_0, with $F'(x_0)=f(x_0)$

* If f not continuous, F may not be differentiable.

Fundamental Thm of Calculus

Let $f\in R$ on $[a,b]$ and $\exists$ differentiable function $F(x)$ on $[a,b]$ s.t. $F'(x)=f(x)$, then $\int_a^b f(x)\,dx=F(b)-F(a)$

Thm Suppose F, G differentiable and F', G' integrable. $f = F'$, $g = G'$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

4.30

Thm Let α monotone increasing on $[a,b]$. Suppose $f_n \in R(\alpha)$ and $f_n \to f$ uniformly on $[a,b]$. Then f is integrable and

$$\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$$

Corollary Suppose $f_n \in R(\alpha)$ on $[a,b]$ and $F(x) = \sum_{n=1}^{\infty} f_n(x)$ $\forall x \in [a,b]$. The series converge uniformly (i.e. $F_N(x) = \sum_{n=1}^{N} f_n(x)$ and $F_N \to F$ uniformly), then $F \in R(\alpha)$ and $\int_a^b F(x)d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n(x)d\alpha$

Thm Suppose (f_n) is a sequence of differentiable functions on $[a,b]$. s.t. $f_n'(x)$ converges uniformly to $g(x)$. and $\exists x_0 \in [a,b]$ s.t. $(f_n(x_0))$ converges.

Then, $f_n(x)$ converges to some function f uniformly, and $f'(x) = g(x) = \lim_{n\to\infty} f_n'(x)$