• Uniform Continuity:

X, Y metric spaces.

Def: $f: X \to Y$ is unif continuous if $\forall \varepsilon > 0$, $\exists \delta > 0$,

s.t. $\forall p, q \in X$, $d_X(p,q) < \delta \Rightarrow d_Y(f(p), f(q)) < \varepsilon$.

Compare with usual def'n of continuity, $f: X \to Y$

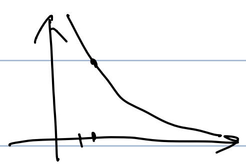
is cont. if $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta(p, \varepsilon) > 0$, s.t.

$\forall q \in X$, $d(q, p) < \delta \Rightarrow d(f(q), f(p)) < \varepsilon$.

Here, in unif cont., one δ works for all $p \in X$.

Ex: • $f(x) = x^2 : \mathbb{R} \to \mathbb{R}$. Continuous but not unif cnt.

• $f(x) = \frac{1}{x} : (0, \infty) \to \mathbb{R}$ continuous but not unif.



$\forall \delta > 0$, $\exists$ $p, q \in (0,1)$, s.t.

$|p - q| < \delta$, and $\left|\frac{1}{p} - \frac{1}{q}\right| > 1$.

• $f(x) = \sin(x) : \mathbb{R} \to \mathbb{R}$ uniformly continuous.

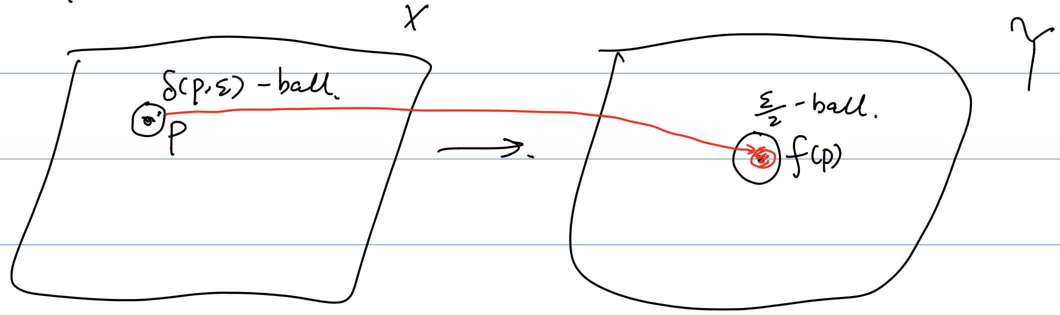
Thm: If $f: X \to Y$ is continuous, and

X is compact, then f is unif cont.

Pf #1 (using "open cover" version of compactness).

Given an $\varepsilon > 0$, $\forall p \in X$, $\exists$ $\delta(p, \varepsilon)$, s.t.

$\forall q \in B_{\delta(p,\varepsilon)}(p)$, $\quad d_Y(f(p), f(q)) < \varepsilon/2$.



- $\forall x, y \in B_{\delta(p)}(p)$, $\quad d_Y(f(x), f(y)) < d_Y(f(x), f(p)) + d_Y(f(y), f(p))$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

- Cover X by balls $B_{\frac{\delta(p)}{2}}(p)$,

$$X = \bigcup_{p \in X} B_{\delta(p)/2}(p).$$

By compactness of X, $\exists\ p_1, \cdots, p_N \in X$, s.t.

$$X = \bigcup_{i=1}^{N} B_{\delta(p_i)/2}(p_i).$$

- Now, let $\underline{\delta} = \min\{\delta(p_i)/2 : i = 1, \cdots, N\}$.

<u>Claim</u> : $\forall \alpha, \beta \in X$, s.t. $d_X(\alpha, \beta) < \delta$, we have

$$d_Y(f(\alpha), f(\beta)) < \varepsilon.$$

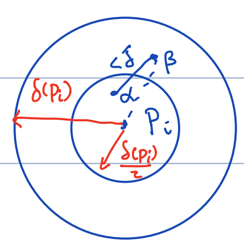
<u>Pf of claim:</u>

Say α is covered by $B_{\delta(p_i)/2}(p_i)$, for some i.

Then β is covered by $B_{\delta(p_i)}(p_i)$, since

$$d(\beta, p_i) \leq d(\beta, \alpha) + d(\alpha, p_i) \leq \delta + \delta(p_i)/2 \leq \delta(p_i)/2 + \delta(p_i)/2$$

$$= \delta(p_i).$$



Now, $\alpha, \beta \in B_{\delta(p_i)}(p_i)$, hence $d_Y(f(\alpha), f(\beta)) < \varepsilon$. #.

<u>Pf #2</u> (using seq compactness of X).

Prove by contradiction. Suppose $\exists\, \varepsilon > 0$, s.t. $\forall \delta > 0$, $\exists\, p, q \in X$, $d(p,q) < \delta$. and $d(f(p), f(q)) > \varepsilon$. Then by taking $\delta = \frac{1}{n}$, $n \in \mathbb{N}$, we get a seq of pairs (p_n, q_n). s.t. $d(p_n, q_n) < \frac{1}{n}$. and $d(f(p_n), f(q_n)) > \varepsilon$.

By compactness of X, we get a subseq of (p_n), say $(p_n)_{n\in A}$. ($A \subset \mathbb{N}$, index subset)., and $\lim\limits_{n\in A} p_n = p$.

With the same index subset.

$$\lim_{n\in A} q_n = p_i$$

$\boxed{\begin{array}{l}\text{Lemma: if } d(s_n, t_n) \to 0 \\ \text{and } \lim t_n = t. \\ \text{then } \lim s_n = t.\end{array}}$

By continuity of f, $\lim\limits_{\substack{n\to\infty \\ n\in A}} f(p_n) = f(p)$

and $\lim\limits_{\substack{n\to\infty \\ n\in A}} f(q_n) = f(p)$, This contradict with.

$$d(f(p_n), f(q_n)) > \varepsilon \qquad \forall n \in A. \qquad \#$$

$\because \exists\, N_1 > 0$, s.t. $\forall n \in A$, $n > N_1$, $d(f(p_n), f(p)) < \frac{\varepsilon}{3}$.

$\because \exists\, N_2 > 0$. s.t. $\forall n \in A$, $n > N_2$, $d(f(q_n), f(p)) < \frac{\varepsilon}{3}$.

$\therefore \forall\, n > \max(N_1, N_2)$, $d(f(p_n), f(q_n)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon$.

contradict with $d(f(p_n), f(q_n)) > \varepsilon$.

Rmk: if $f: X \to Y$ unif cont., and $S \subset X$ with the induced metric, then

$$f|_S : S \to Y$$

is also uniformly continuous.

Rmk: if $f: X \to Y$ cont, $S \subset X$, then $f|_S : S \to Y$ is cont.

Ex: $f(x) = \frac{1}{x} : [\frac{1}{2}, 2] \to \mathbb{R}$.

it is continuous, and domain is cpt hence is uniformly continuous.

Thm: $f: X \to Y$ cont., X is compact, and f is a bijection. then $f^{-1}: Y \to X$ is continuous.

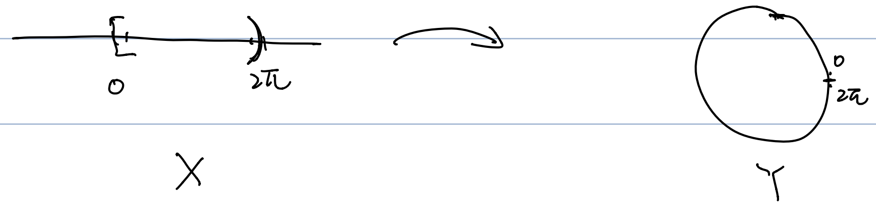
Pf: . denote $h = f^{-1}$. Then $\forall y \in X$, exist a unique $\exists! x \in X$, s.t. $f(x) = y$. $h(y) = x$. To show that h is cont. just need to show $h^{-1}(\text{open})$ is open., which is equiv to $h^{-1}(\text{closed set})$ is closed. . $\forall E \subset X$, closed, we need to show $h^{-1}(E)$ is closed. $\because h^{-1}(E) = f(E)$. and since E is closed in a compact set. X, hence E is compact, hence. $f(E)$ is compact, hence $f(E)$ is closed. Thus, $h^{-1}(E)$ is closed. Thus, h is continuous. #.

Ex: if X is not compact, then conclusion can fail.

$X = [0, 2\pi)$, $\quad Y = S^1$ unit circle in $\mathbb{R}^2$.

$f : X \to Y$. $\quad \theta \mapsto (\cos\theta, \sin\theta)$.

Hence $f^{-1} : Y \to X$ is not continuous.



Connectedness and Continuity:

• <u>Def</u>: Let X be a topological space. X is <u>connected</u> if and only if the only subset of X that is both open and closed are X and ϕ.

• X is not connected $\Leftrightarrow$ $\exists\, U, V \subset X$, non-empty, open, $U \cap V = \phi$, and $X = U \amalg V$.

$\Rightarrow$ X is not conn., meaning $\exists\, S \subset X$, $S \neq \phi$, $S \neq X$, S is both open and closed,

then $X = \underset{\text{open}}{S} \amalg \underset{\text{open}}{S^c}$.

• <u>Thm</u>: If $f : X \to Y$ is continuous, X is connected, then $f(X)$ is connected. (as subset of Y, with <u>induced</u> <u>topology</u>)

<u>Pf</u>: Recall, a subset $E \subset f(X)$ is open, if and only if $\exists\, \tilde{E} \subset Y$, open. s.t. $\tilde{E} \cap f(X) = E$.

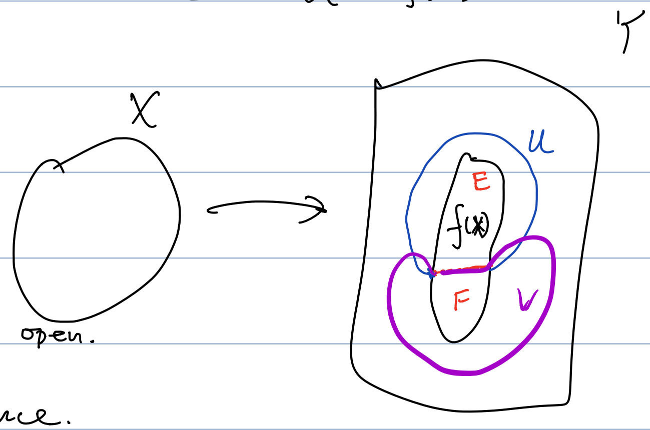
So, suppose $f(X)$ is not connected. Then.

$f(X) = E \amalg F$, E, F open disjoint subsets in $f(X)$.

ie. $\exists\, U, V$ open in Y, s.t. $E = U \cap f(X)$

and $F = V \cap f(X)$.

Then., $X = f^{-1}(E) \amalg f^{-1}(F)$.



$f^{-1}(E) = f^{-1}(U \cap f(X))$

$= f^{-1}(U)$. i.e. $f^{-1}(E)$ is open.

similarly, $f^{-1}(F)$ is open. Hence.

X is disconnected, and we get a contradiction.

Cor: if $f: X \to Y$ is cont. $E \subset X$, is connected, then $f(E)$ is conn.

Pf: Consider $f|_E : E \to Y$. and apply the thm.

$\equiv$.

Real valued function on $\mathbb{R}$.

$f: \mathbb{R} \to \mathbb{R}$.

Prop: $[0,1] \subset \mathbb{R}$ is connected.

Pf: • Suppose $[0,1]$ is the disjoint union of 2 open non-empty subset U, V in $[0,1]$. (this doesn't mean, U, V are open as subset of $\mathbb{R}$, this only means, $U = \tilde{U} \cap [0,1]$, $V = \tilde{V} \cap [0,1]$, for some $\tilde{U}, \tilde{V} \subset \mathbb{R}$ open)

• U, V are closed in $[0,1]$. $\Rightarrow$ U, V are also closed in $\mathbb{R}$.

• Let $a \in U$, and $b \in V$. W.L.O.G, assume $a < b$. Consider $U \cap [a,b]$., and let

. $x = \sup(U \cap [a,b])$.

$\exists\ (x_n)$ in $U \cap [a,b]$, s.t. $x_n \to x$, $\Rightarrow$ $x \in U$ since U is closed.

. $U = \tilde{U} \cap [0,1]$.

$U \cap [a,b] = \tilde{U} \cap [0,1] \cap [a,b] = \tilde{U} \cap [a,b]$.

hence $x = \sup(\tilde{U} \cap [a,b]) \notin \tilde{U}$, This is a contradiction. #.