0. Review

- Metric space (X, d)
- Topology of a metric space. (X, d)

open ball
radius $= \delta$.
center $= p$.

$U \subset X$ is open. if $\forall p \in U$, $\exists \delta > 0$. s.t. $B_\delta(p) \subset U$.

- Compact subset.

Def: $K \subset X$ is compact, if $\forall$ open cover of K,

$\exists$ a finite subcover.

Prop:
- K compact $\Rightarrow$ K bounded.
- K compact $\Rightarrow$ K is closed.
- $E \subset X$ closed, K compact, $E \subset K \Rightarrow E$ is also compact.

$\forall$ metric space,

Thm 1: compactness $\Leftrightarrow$ sequential compactness.

K compact $\Leftrightarrow$ $\forall$ sequence (x_n) in K, $\exists x \in K$, and a subseq $(x_{n_k})_k$ such. $x_{n_k} \to x$.

(Heine-Borel). $\Rightarrow$

Thm 2: In $\mathbb{R}^n$, K compact $\Leftrightarrow$ K closed and bounded.

Rudin Thm 2.41

Continuous Map: Let (X, d_X), (Y, d_Y) be metric spaces.

$f: X \to Y$.

Def 1: f is cont. iff $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$f(B_\delta(p)) \subset B_\varepsilon(f(p)).$$

Def 2: f is cont. iff $\forall\ V \subset Y$ open, $f^{-1}(V)$ is open.

Def 3: f is cont. iff $\forall\ x_n \to x$ in X, we have $f(x_n) \to f(x)$ in Y.

Rmk: ① The notion of "open" and "closed" depends on the ambiant space, the notion of "compact" is intrinsic. $d_X: X \times X \to \mathbb{R}$

Let (X, d_X) be a metric space. Let $S \subset X$. Then $(S, \underbrace{d_X|_{S\times S}}_{d_S})$ is also a metric space. A subset $V \subset S$, If V is open in S, V may not be open in X.

Ex: • $X = \mathbb{R}$, $S = (0, 1)$.

Consider $E = (0,1) \subset S$.

Then: • E is open in S

• E is closed in S

• E is open in X

• E is not closed in X.

But E is not closed in X.

• $X = \mathbb{R}$, $S = \{0\}$ 2 metric spaces.

• $E = \{0\}$. Then

• E is open and closed in S

• E is closed in X,

• E is not open in X.

$\subsetneq$

$\subset$ $\subsetneq$

Ex: • (X, d) metric space. $S \subset X$. (S, d) metric space with induced metric.

• $K \subset X$ is compact, $K \subset S$.

$\Rightarrow$ K is also compact as a subset of S.

$\ulcorner$ $\because$ $\forall\, \mathcal{U} \subset S$ open in S, $\exists\, \tilde{\mathcal{U}} \subset X$, open in X, s.t.

$\llcorner$ $\mathcal{U} = S \cap \tilde{\mathcal{U}}$.

Hence if $K \subset \bigcup_\alpha \mathcal{U}_\alpha$, $\mathcal{U}_\alpha$ open in S.

then we promote each $\mathcal{U}_\alpha$ to an open subset $\tilde{\mathcal{U}}_\alpha \subset X$.

s.t. $\mathcal{U}_\alpha = \tilde{\mathcal{U}}_\alpha \cap S$. Then.

$$K \subset \bigcup_\alpha \tilde{\mathcal{U}}_\alpha.$$

$\because$ K is compact in X. $\therefore$ $\exists\, \{\alpha_1, \cdots, \alpha_n\}$ finite index subset.

s.t. $K \subset \tilde{\mathcal{U}}_{\alpha_1} \cup \tilde{\mathcal{U}}_{\alpha_2} \cup \cdots \cup \tilde{\mathcal{U}}_{\alpha_n}$

intersect both sides with S $\Rightarrow$.

$$K \subset \mathcal{U}_{\alpha_1} \cup \mathcal{U}_{\alpha_2} \cup \cdots \cup \mathcal{U}_{\alpha_n}.$$

Hence K is compact subset in S.

(we can take $S = K$.)

2. (X, d_X), (Y, d_Y), metric space

$f: X \to Y$.

Assume: X is a compact metric space.

Ex: • $X = [0, 1]$, (with the induced metric from $\mathbb{R}$)

X is compact.

• $X = (0,1)$, X is <u>not</u> a compact metric space.

e.g. the seq. $x_n = \frac{1}{n}$. has no convergent subseq. in X.

Let X, Y be metric spaces (not assuming X is compact).

<u>Thm</u>: $f: X \to Y$, continuous. if $E \subset X$ is compact, then $f(E) \subset Y$ is compact.

<u>Pf</u>: We need to show, for any open cover of $f(E)$, there exists a finite subcover.

Let $f(E) \subset \bigcup_{\alpha \in A} U_\alpha$, $U_\alpha \subset Y$ open.

Then. $E \subset f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$.

Since f is cont, hence $f^{-1}(U_\alpha)$ is open $\forall \alpha \in A$. By compactness of E, there is a finite subset $I \subset A$. s.t.

$$E \subset \bigcup_{\alpha \in I} f^{-1}(U_\alpha).$$

Apply f to both sides, we get

$$f(E) \subset f\left(\bigcup_{\alpha \in I} f^{-1}(U_\alpha)\right) = \bigcup_{\alpha \in I} U_\alpha. \quad \#.$$

<u>Cor</u>: If $f: X \to \mathbb{R}$ continuous, X is compact, then $\exists p, q \in X$, s.t.

$$f(p) = \max(f(X)), \quad f(q) = \min(f(X)).$$

Remark: $K \subset \mathbb{R}$, K is compact, then.

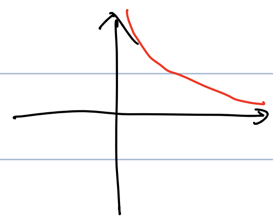
$\sup(K) \in K$. $\inf(K) \in K$.

Hence $\max(K)$, $\min(K)$ exist. equals to sup & inf, resp.

Rmk: • If $f: X \to Y$ is cont., f send compact set in X to compact set in Y. But, given $E \subset Y$ compact $f^{-1}(E)$ is not guaranteed to be compact.

f^{-1}(compact) may not be compact.

Ex: (1) $f: (0,\infty) \to \mathbb{R}$. $f(x) = \frac{1}{x}$.



$f^{-1}([0,1]) = [1,\infty)$. closed, but not bounded.

Q: is $[1,\infty)$ closed? ✓ yes.

$\Leftrightarrow$ is the complement $(-\infty,1)$ open? ✓ yes.

$\forall p \in (-\infty,1)$, $\exists B_\delta(p) \subset (-\infty,1)$. $\Rightarrow (-\infty,1)$ is open

f continuous $\Leftrightarrow$ f^{-1}(open) is open

$\Leftrightarrow$ f^{-1}(closed) is closed.

(2): $f: (0,1) \to \mathbb{R}$ inclusion map. (continuous).

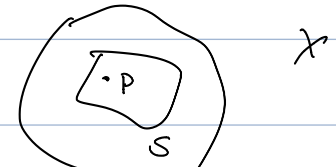
$x \mapsto x$.

$f^{-1}([\frac{1}{2},\frac{3}{2}]) = [\frac{1}{2},1)$. is this a compact set?

closed & bounded, as subset of $(0,1)$.

(X, d) metric space. $S \subset X$, with induced metric.

$\iota : S \to X$



Is the inclusion map continuous?

Yes. $\forall p \in S$, $\forall \varepsilon > 0$, we ~~set~~ set $\delta = \varepsilon$.

indeed. $\iota(B_\varepsilon^{(S)}(p)) \subset B_\varepsilon^{(X)}(\iota(p))$.

$B_\varepsilon^{(S)}(p) = \{ x \in S \mid d(x,p) < \varepsilon \}$ $\qquad d_X = d_S = d$

$B_\varepsilon^{(X)}(\iota(p)) = \{ x \in X \mid d(x,p) < \varepsilon \}$. $\qquad \iota(p) = p$.

$\iota(B_\varepsilon^{(S)}(p)) \subset B_\varepsilon^{(X)}(\iota(p))$.

? $(0, \frac{1}{2})$, is it open in $\underbrace{(0,1)}_{S}$? Yes

- open set in $\underset{S}{(0,1)}$:

Q: $\forall x \in (0, \frac{1}{2})$, $\exists \varepsilon > 0$, $B_\varepsilon^{(S)}(x) \subset (0, \frac{1}{2})$ ✓ ?

$\because$ set $\varepsilon = \frac{1}{2}\min(x, \frac{1}{2} - x)$

$B_\varepsilon^{(S)}(x) = \{ y \in S \mid d(y, x) < \varepsilon \}$.

e.g. $B_{\frac{1}{4}}^{(S)}\frac{1}{4} = \{ y \in (0,1) \mid |y - \frac{1}{4}| < \frac{1}{4} \} = (0,1) \cap (0, \frac{1}{2})$

$= (0, \frac{1}{2})$.

(X, d) metric space. $S \subset X$ open subset.

- $U \subset S$ is an open subset in S.

iff U is an open subset in X.

Ex: open cover of $\underset{S}{(0,1)}$.

- $(0,1) = (0, \frac{1}{2}) \cup (\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{2}, 1)$
- $(0,1) = \left(\bigcup^{\infty} B_{\frac{1}{2n}}(\frac{1}{n}) \right) \cup \left(\bigcup^{\infty} B_1(1 - \frac{1}{n}) \right)$

$n=2$ $\qquad$ $n=2$ $\quad 2n.$