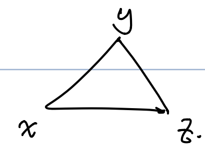
Today: Review for midterm 2.

- Series and convergences. (ratio test, root test, integral test. alternating series test).
- Metric space and topology.
- continuous maps between metric space.
  - uniform continuity.
- Convergence of a sequence of functions.

Metric Space: A set X, together with a distance function.

$d(\cdot,\cdot): X \times X \to \mathbb{R}$. such that ① ② ③ triangle in-equality.

$d(x,y) + d(y,z) \geq d(x,z) \quad \forall x, y, z \in X$.



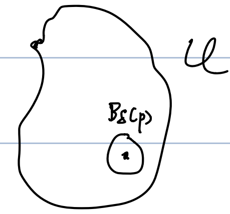
Induced metric on a subset: (X, d_X) a metric space.

$S \subset X$ subset. we define $d_S: S \times S \to \mathbb{R}$.

$d_S(x,y) = d_X(x,y) \quad \forall x, y \in S$.

Topology of a metric space: (X, d_X) a metric space.

For a subset $U \subset X$. we say U is open, if and ~~only~~ if $\forall p \in U$. $\exists \delta > 0$, s.t. $B_\delta(p) = \{x \in X \mid d(p,x) < \delta\}$ (the open ball) is contained in U.



We verified that, the collection of open subsets in X, satisfies the 3 axioms:

① X, $\emptyset$ are open.

② The union of an arbitrary collection of open sets is open

③ The intersection of finitely many open subset is open.

$\rightsquigarrow$ we get a topology on X.

Closed set : $E \subset X$ is closed iff E^c is open.

Rmk : • it is possible to have subset both open and closed, e.g. X, and ϕ.

• "open" and "closed" are relative notion, we should say a subset S of a (ambiant) topological space X is open or closed.

Compact set : • Let X be a topological space. $K \subset X$ is compact iff $\forall$ open cover of K, there exists a finite subcover.

• (Sequential Compactness) Let X be a metric space. $K \subset X$ is seq. compact, if $\forall$ sequences in K, there exists a convergent subsequence (with the limit point also in K).

• compactness $\Longleftrightarrow$ sequential compactness.

• (Heine-Borel) : A subset $K \subset \mathbb{R}^n$ is compact

$\Leftrightarrow$ K is closed and bounded.

in general, (in a metric space)

- Compactness $\Rightarrow$ closed, bounded,

closed subset of compact set is compact.

- Compact set is an "absolute" notion, we can say " K is a compact set " without specifying the ambiant space.

- Connectedness : A top. space X is connected iff X cannot be written as a disjoint union of two non-empty open subsets.

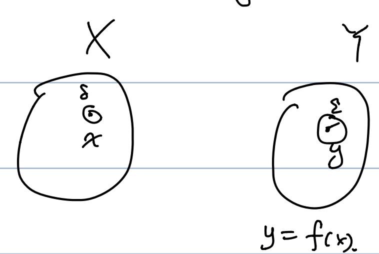
$E \subset \mathbb{R}$ is connected. $\Leftrightarrow$ $\forall\, x, y \in E$, such that $x < y$, we have $[x, y] \subset E$.

$\Rightarrow$ connected set in $\mathbb{R}$ are of the form (a, b), $[a, b)$, $(a, b]$, $[a, b]$, ...

These were in Rudin Ch2.

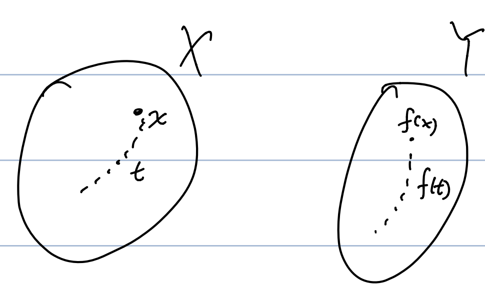
Continuous Maps : Let $f: (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. Three def'n for continuity

Def 1 : $\forall\, x \in X$, $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, s.t. $f(B_\delta(x)) \subset B_\varepsilon(f(x))$.



Def 2 : $\forall x \in X$, a limit point of X, we require.

$$\underbrace{\lim_{t \to x} f(t)}_{\text{notion of limit of a function}} = f(x).$$



Def 3 : For any open subset $V \subset Y$, $f^{-1}(V)$ is an open subset of X.

Property : (1) if $f: X \to Y$ cts, $K \subset X$ compact, then $f(K)$ is compact.

(2). if $f: X \to Y$ cts, $K \subset X$ is connected, then $f(K)$ is connected.

Uniform Continuity (a concept for map between metric spaces).

$f: (X, d_X) \to (Y, d_Y)$ is uniformly continuous, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x, y \in X$, if $d(x,y) < \delta$, then $d(f(x), f(y)) < \varepsilon$.

Ex : (non. unif continuous function) : $f(x) = x^2$ on $\mathbb{R}$.

or $f(x) = \sin(\frac{1}{x})$ $x \in (0, \infty)$.

Prop : If $f: X \to Y$ cts, X is compact, then f is unif continuous.

Ch 4 of Rudin

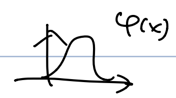
- Sequences of function and convergence:

Let $f_n : X \to Y$ be a sequence of fcn. between metric space. Let $f : X \to Y$ be a fcn. We say $f_n \to f$ $\boxed{\text{pointwise}}$, if $\forall x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x). \iff \lim_{n\to\infty} |f_n(x) - f(x)| = 0.$$

We say $f_n \to f$ $\boxed{\text{uniformly}}$, if

$$\lim_{n\to\infty} \Big(\underbrace{\sup_{x\in X} |f_n(x) - f(x)|}_{d_\infty(f_n, f)} \Big) = 0.$$



☆ Examples about the difference. running bumps. squeezing bump.

- $f_n(x) = \varphi(x-n)$. a bump running to the right.

  $f_n \to 0$ ptwise. but not uniformly.

- $f_n(x) = \varphi(x \cdot n)$ $f_n \to 0$ ptwise, but not uniformly.

Ch 7 of Rudin (first 3 sections).

Problems :

(1). A metric space (X, d) is said to satisfy "ultra metric" property. if $\forall x, y, z \in X$.

$$d(x,z) \le \max\{d(x,y), d(y,z)\}.$$

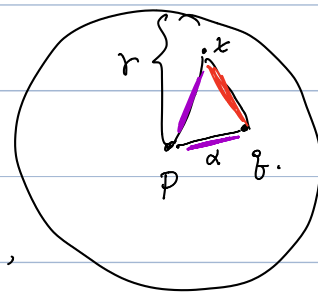
stronger notion of triangle ineq.

(compare with $d(x,z) \le d(x,y) + d(y,z)$.)

Prove that: If $q \in B_r(p)$, then

$$B_r(p) = B_r(q).$$

i.e. any point in a given open ball can be used as the center.



pf: We need to show $B_r(p) \subset B_r(q)$ and $B_r(p) \supset B_r(q)$. For any $x \in B_r(p)$, $d(x,p) < r$.

Then $\quad d(x,q) \le \max\{d(p,q), d(x,p)\}$.

since $d(p,q) < r$, $d(x,p) < r$. hence max of the 2 $< r$.

Then $\quad d(x,q) < r$

For any $x \in B_r(q)$, we have $d(x,q) < r$.

Then $\quad d(x,p) \le \max\{d(x,q), d(q,p)\} < r$.

Ex of a ultra metric space:

$X = \mathbb{Q}$. Fix $p > 0$ a prime number,

Then any $x \in \mathbb{Q}$, can be written as

$$x = p^n \cdot \frac{\alpha}{\beta} \qquad \alpha \in \mathbb{Z},\ \beta \in \mathbb{N}.$$

α, β has no common factor

and no factors of p.

We define a "norm" $\|x\|_p := \frac{1}{p^n}$, $\|0\| = 0$.

Ex $p=3$, $\|5\|_p = 1$, $\|3\| = \frac{1}{3}$,

$\|3^2\|_p = \frac{1}{3^2}$, $\|\frac{1}{3^2} \cdot \frac{17}{25}\|_p = 3^2$.

Claim: $\forall\, x, y \in \mathbb{Q}$, we have

$$\max\{\|x\|_p, \|y\|_p\} \geq \|x+y\|_p.$$

Pf: If $\|x\|_p > \|y\|_p$, i.e. $\|x\|_p = \frac{1}{p^n}$, $\|y\|_p = \frac{1}{p^m}$,

$\underline{\underline{n<m}}$, then

$$x = p^n \cdot \frac{\alpha}{\beta}, \quad y = p^m \cdot \frac{\alpha'}{\beta'}$$

$$x+y = p^n \left\{ \frac{\alpha}{\beta} + p^{m-n} \frac{\alpha'}{\beta'} \right\}$$

$$= p^n \cdot \frac{\alpha \cdot \beta' + \beta \cdot p^{m-n} \cdot \alpha'}{\beta \cdot \beta'}$$

both denominator and numerator are not divisible by p.

$\therefore$ $\|x+y\|_p = \frac{1}{p^n}$, hence

$$\|x+y\|_p = \max\{\|x\|_p, \|y\|_p\}.$$

if $\|x\|_p = \|y\|_p = \frac{1}{p^n}$, leave as exercise. #

Given the claim, we can define a metric on $\mathbb{Q}$.

$$d(x,y) := \|x-y\|_p.$$

Then, from the claim, we have,

$$d(x,z) \le \max\{d(x,y), d(y,z)\} \quad \forall x,y,z \in \mathbb{Q}$$

i.e. $\|x-z\| \le \max\{\|x-y\|_p, \|y-z\|_p\}$.

$\equiv$

<u>Ex 2</u>: Let X be a complete metric space (e.g. $X = \mathbb{R}$).

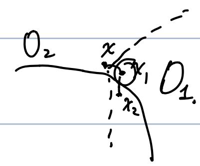
Let $O_n \subset X$ be a sequence of open and dense subset.

Prove that $\bigcap_{n=1}^{\infty} O_n$ is still dense.

<u>Ex 2 baby</u>: $\bigcap_{n=1}^{N} O_n$ still open and dense.

<u>sketch</u>: Let $x \in X$ be any point. We need to show that,

$\forall \varepsilon > 0$, $B_\varepsilon(x) \cap (\bigcap_{n=1}^{\infty} O_n) \neq \emptyset$.

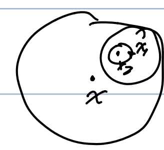


$x_1 \in O_1$

$x_2 \in O_1 \cap O_2$

$x_3 \in O_1 \cap O_2 \cap O_3$.

$\vdots$



(x_i) Cauchy $\Rightarrow$ x_i converges to z.

show that $z \in \bigcap_{n=1}^{\infty} O_n$.

$E \subset X$ is dense, if $\overline{E} = X$.

$\Leftrightarrow \forall x \in X, \ \forall \delta > 0, \ B_\delta(x) \cap E \neq \emptyset$.