· Function between Metric Space. <u>Rudin</u>, §4. §7.

<u>Recall</u>: A function $f: A \to B$ from a set A to a set B, is an assignment, to every $\alpha \in A$, an element $f(\alpha) \in B$.

- domain of f: A (source)
- range of f: $f(A) \subset B$. (target)

· f is injective, if $\forall x, y \in A$, and $x \neq y$, then $f(x) \neq f(y)$.

· f is surjective, if $f(A) = B$, i.e. $\forall \beta \in B$, $\exists \alpha \in A$, s.t. $f(\alpha) = \beta$.

· $\forall E \subset B$. let $f^{-1}(E) = \{\alpha \in A \mid f(\alpha) \in E\}$.

<u>Lemma</u>:

Let $A' \subset A$, $B' \subset B$, $f: A \to B$. then.

$$f(A') \subset B' \iff A' \subset f^{-1}(B').$$

pf:
$$\begin{aligned} f(A') \subset B' &\iff \forall x \in A', \quad f(x) \in B' \\ &\iff \forall x \in A', \quad x \in f^{-1}(B') \\ &\iff A' \subset f^{-1}(B'). \end{aligned}$$

· Let (X, d_X) and (Y, d_Y) be metric spaces.

<u>Def</u>: A function $f: X \to Y$ is continuous at $p \in X$, if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$\forall x \in X$, with $d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$.

$\Updownarrow$ equivalent to

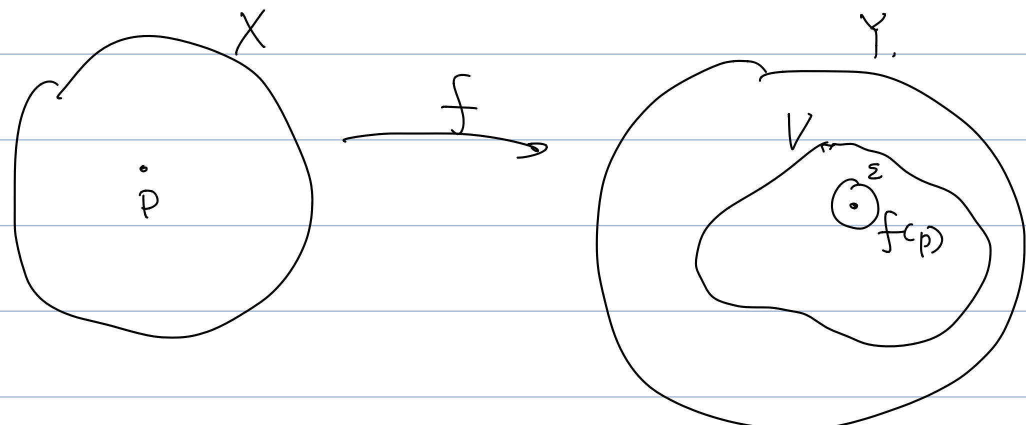
$$f(B_\delta(p)) \subset B_\varepsilon(f(p)).$$

X, Y as before.

Thm. A function $f: X \to Y$ is continuous, iff $\forall V \subset Y$ open, $f^{-1}(V)$ is open.

rmk: it only uses the notion of open sets, works for general topological spaces.

Pf: $\Rightarrow$ suppose f is continuous, then we need to show $\forall V \subset Y$ open, $f^{-1}(V)$ is open. i.e. $\forall p \in f^{-1}(V)$, we need to show $\exists \delta > 0$, s.t. $B_\delta(p) \subset f^{-1}(V)$.



Since $f(p) \in V$, and V is open, we have $\varepsilon > 0$, and $B_\varepsilon(f(p)) \subset V$. By continuity of f, $\exists B_\delta(p)$, s.t. $f(B_\delta(p)) \subset B_\varepsilon(f(p))$. Hence

$$f(B_\delta(p)) \subset B_\varepsilon(f(p)) \subset V \Rightarrow B_\delta(p) \subset f^{-1}(V).$$

Hence $f^{-1}(V)$ is open.

$\Leftarrow$. If $f^{-1}(V)$ is open for any $V \subset Y$ open., we need to show that $\forall p \in X$, $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t. $f(B_\delta(p)) \subset B_\varepsilon(f(p))$, Since $B_\varepsilon(f(p))$ is open, hence $f^{-1}(B_\varepsilon(f(p)))$ is open, and contains p. By definition of open set. $\exists \delta > 0$. s.t.

$$B_\delta(p) \subset f^{-1}(B_\varepsilon(f(p)))$$

$$\Leftrightarrow \quad f(B_\delta(p)) \subset B_\varepsilon(f(p)).$$

\#.

<u>Def</u> (<u>limit of a function</u>). Let X, Y be metric spaces. Let $E \subset X$ be a subset ($(E, d_X|_E)$ is a metric space). and $f: E \to Y$. Suppose $\underline{p}$ is a limit point of E, then we say $\lim_{x \to p} f(x) = q$, if there is a point $q \in Y$. such that $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t.

$$f(B_\delta^X(p) \cap E) \subset B_\varepsilon(q).$$

$B_\delta^X(p) = \{x \in X \mid x \neq p,\ d(x,p) < \delta\}$.

i.e. $\forall x \in E$, s.t. $0 < d_X(x,p) < \delta$, we have.

$$d_Y(f(x), q) < \varepsilon.$$

<u>Rmk</u>:

- P is a limit point of E, if $\forall \delta > 0$,
$$B_\delta^X(p) \cap E \neq \emptyset.$$

- E' is the set of limit points of E.

eg $E = (0,1)$, $E' = [0,1]$.
$E = \{\frac{1}{n}, n \in \mathbb{N}\}$, $E' = \{0\}$.

- $E = E^{\text{isolated}} \sqcup E'$

$\{x \in E \mid \exists \varepsilon > 0,\ B_\varepsilon(x) \cap E = \{x\}\}$

Thm: with (X, Y, E, f) as above, $p \in E'$.

We have $\lim_{x \to p} f(x) = q$. if and only if.

$\forall$ any convergent seq $p_n \to p$ with $p_n \in E$, $p_n \neq p$.

$\lim_{n \to \infty} f(p_n) = q$.

$p_n \neq p$.

Pf: $\Rightarrow$ Suppose $\lim_{x \to p} f(x) = q$. And suppose $p_n \to p$, $p_n \in E$, we need to show $\lim_{n \to \infty} f(p_n) = q$. For any $\varepsilon > 0$, we need to have ~~any~~ an $N > 0$. s.t. $\forall n > N$. $d(f(p_n), q) < \varepsilon$. By definition of a limit of function, $\exists \delta > 0$, s.t. if $d_X(p_n, p) < \delta$, then. $d(f(p_n), q) < \varepsilon$. By $p_n \to p$, $\exists N > 0$, s.t. $\forall n > N$, $d(p_n, p) < \delta$. Hence, in summary, $\exists N > 0$. s.t. $\forall n > N$, $d(p_n, p) < \delta \Rightarrow d(f(p_n), q) < \varepsilon$.

$\Leftarrow$ Suppose $\lim_{x \to p} f(x) \neq q$. that means. $\exists \varepsilon > 0$, s.t. $\forall \delta > 0$,

$$f(B^\times_\delta(p) \cap E) \text{ is not contained in } B_\varepsilon(q).$$

i.e. $\exists x \in E$, s.t. $0 < d_X(x, p) < \delta$, s.t.

$$d(f(x), q) > \varepsilon.$$

Let δ take values $\frac{1}{n}$, for $n \in \mathbb{N}$, and one obtain a sequence of pts x_n, s.t. $0 < d(x_n, p) < \frac{1}{n}$, $x_n \in E$, and $d(f(x_n), q) > \varepsilon$. This contradict with the statement that for all seq $x_n \to p$. $x_n \neq p$, $f(x_n) \to q$. #

To show statements $P \Leftrightarrow Q$,

we show $P \Rightarrow Q$

and $!P \Rightarrow !Q$ (P is not true $\Rightarrow$ Q is not true)

$f+g : X \to \mathbb{R}$

$(f+g)(x) := f(x) + g(x)$.

(4.4).

Thm: Let $f, g : X \to \mathbb{R}$. And assume that

$\lim_{x \to p} f(x) = A$, $\lim_{x \to p} g(x) = B$.

Then (1) $\lim_{x \to p} \underbrace{(f+g)}(x) = A + B$

(2) $\lim_{x \to p} (f \cdot g)(x) = A \cdot B$

(3) if $B \neq 0$, $g(x) \neq 0$ $\forall x \in X$, then

$\lim_{x \to p} (f/g)(x) = A/B$.

Pf: using the corresponding result about sequences. #

(3rd def of continuity)

Thm: $f : X \to Y$. f is continuous, if and only if for any

$p \in X'$, a limit pt of X, we have

$$f(p) = \lim_{x \to p} f(x).$$

(i.e. $f(\lim_{x \to p} x) = \lim_{x \to p} f(x)$).

Operations on Continuous Functions.

Thm: Let $f, g : X \to \mathbb{R}$ be continuous functions, then $f+g$, $f \cdot g$, f/g (if $g \neq 0$) are continuous functions.

Pf: To show $f+g$ is continuous, just need to show $\forall\, p \in X'$,

$$\lim_{x \to p} f(x) + g(x) = f(p) + g(p).$$

This follows from Thm 4.4 above. - - - #.

Thm: if $f : X \to Y$, $g : Y \to Z$ are continuous then $(g \circ f) : X \to Z$ is continuous.

[$(g \circ f)(x) = g(f(x))$ is the composition of f, g]

Pf: Just need to show that, $\forall\, V \subset Z$ open. $(g \circ f)^{-1}(V)$ is open. But.

$$(g \circ f)^{-1}(V) = f^{-1}(\underbrace{g^{-1}(V)}_{\text{open} \because g \text{ cont.}}) \text{ is open. \#.}$$

$\boxed{\text{Facts}}$: If X, Y are topological space, then $X \times Y$ is a topological space, with open sets "generated" by $U \times V$, $U \subset X$ open, $V \subset Y$ open.

↑ using $\cup, \cap, \cdots$

Given ~~a~~ maps $f: Z \to X$, $g: Z \to Y$.

$(f,g): Z \to X \times Y$. $(f,g)(z) = (f(z), g(z))$.

(f,g) is continuous if and only if f and g are continuous.

Thm: Let $f: X \to \mathbb{R}^n$, with $f(x) = (f_1(x), \cdots, f_n(x))$.

Then f is continuous $\Leftrightarrow$ $f_i: X \to \mathbb{R}$ are continuous $\forall i = 1, \cdots, n$.

Pf: See Rudin.

Ex: (1). $x: \mathbb{R} \to \mathbb{R}$ (identify) is continuous.

· $x^2: \mathbb{R} \to \mathbb{R}$ $\because$ multiplication of cont. fcn is cont.

⋮

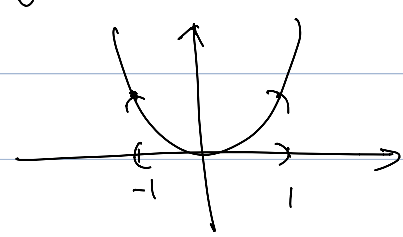
$\Rightarrow x^n: \mathbb{R} \to \mathbb{R}$ cont.

$\Rightarrow p(x) = a_n x^n + \cdots + a_0 : \mathbb{R}^n \to \mathbb{R}$ is continuous.
polynomial

T/F: (1) if $f: X \to Y$ is continuous, then $\forall U \subset X$ open, $f(U)$ is also open in Y.

✗ false.

$f(x) = x^2 : \mathbb{R} \to \mathbb{R}$



$f((-1,1)) = [0, 1)$

(2). if $f: X \to Y$ cont. Then $\forall E \subset Y$ closed, $f^{-1}(E)$ is closed.

<u>True</u>: E is closed $\Leftrightarrow$ E^c is open.

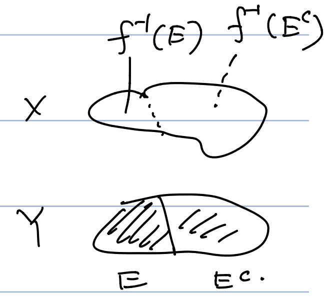
$Y = E \sqcup E^c$.

$f^{-1}(Y) = f^{-1}(E) \sqcup f^{-1}(E^c)$

$f^{-1}(Y) = X$.

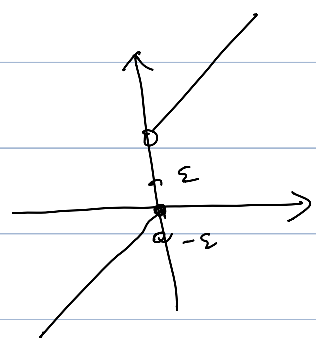
$\Rightarrow \quad X = f^{-1}(E) \sqcup f^{-1}(E^c)$

$\Leftrightarrow \quad \underbrace{f^{-1}(E^c)}_{\text{open}} = \left(f^{-1}(E)\right)^c$



hence $f^{-1}(E)$ is closed.

- <u>Ex</u>:



$f: \mathbb{R} \to \mathbb{R}$.

$f^{-1}\left(\underline{(-\varepsilon, \varepsilon)}\right) = (-\varepsilon, 0]$