· **Def**: Let $f_n : X \to \mathbb{R}$ be seq of fcn.

we say $f_n \to f$ uniformly, if $\forall \varepsilon > 0$, $\exists N > 0$.

s.t. $\forall n > N$, $\forall x \in X$, we have

$$|f_n(x) - f(x)| < \varepsilon.$$

(uniform means. N only deps on ε, not on x.)

· **Equiv Def**: Recall $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$.

$$f_n \to f \text{ unif} \iff \lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

· Just as for seq of numbers (x_n) in $\mathbb{R}$,

(x_n) is convergent $\iff$ (x_n) satisfies Cauchy condition.

$\Rightarrow$ automatic, by triangle inequality

$\because |x_n - x_m| \le |x_n - x| + |x_m - x|$

$\Leftarrow$ dep on that $\mathbb{R}$ is a complete metric space.

· **Thm** (Unif. Cauchy $\iff$ Unif converge)

· Let $f_n : X \to \mathbb{R}$ be a seq of fcn. We say (f_n) is unif. Cauchy, if $\forall \varepsilon > 0$, $\exists N \geq 0$, s.t. $\forall n, m > N$, we have

$$|f_n(x) - f_m(x)| < \varepsilon. \qquad \forall x \in X.$$

- (f_n) is Unif Cauchy $\Leftrightarrow$ (f_n) is unif convergent.

<u>Pf</u>: $\Rightarrow$. Let $\varepsilon > 0$ be given. Pick N, as guaranteed by unif Cauchy, s.t. $\forall\, n, m > N$, we have.

$$(*) \qquad |f_n(x) - f_m(x)| \leq \varepsilon.$$

- Since $\forall x \in X$, $\{f_n(x)\}$ is a Cauchy seq of numbers, hence $f(x) := \lim_{n\to\infty} f_n(x)$ exists. (using the Cauchy $\Leftrightarrow$ conv for seq of #)

- Take $(*)$, and take limit that $m \to \infty$.

  since $\lim_{m\to\infty} f_m(x) = f(x)$, hence

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)|.$$

[if $\lim_n a_n = a$, then $\lim_n C - a_n = C - a$,

then $\lim_{n\to\infty} |C - a_n| = |C - a|$.].

Hence, $|f_n(x) - f(x)| \leq \varepsilon$. $\forall\, n > N$, $\forall x \in X$.

Thus $f_n \to f$ uniformly.

$\Leftarrow$ use triangle ineq. #.

<u>Notation</u>: We say a series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly to f, if the partial sum $F_N = \sum_{n=1}^{N} f_n$

converges uniformly to f.

<u>Thm</u> (Weierstrass M-test).

Suppose $f_n : X \to \mathbb{R}$ is a seq of fcn,

and $0 \le M_n \in \mathbb{R}$, s.t. $M_n \ge \sup_{x \in X} |f_n(x)|$.

If $\sum_{n=1}^{\infty} M_n < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

$\boxed{\text{sketch Pf:}}$ $\left| \sum_{n=N_1}^{N_2} f_n(x) \right| \le \sum_{n=N_1}^{N_2} |f_n(x)| \le \sum_{n=N_1}^{N_2} M_n$.

Hence, Cauchy test for $\sum M_n \Rightarrow$ unif Cauchy test for $\sum_n f_n$.

#

Unif convergence preserves continuity:

Let X be a metric space.

<u>Thm</u>: Let $E \subset X$. Let $f_n : E \to \mathbb{R}$ be a seq of continuous functions. Suppose $f_n \to f$ uniformly on E.

Let $x \in E'$ be a limit point, and assume $A_n = \lim_{t \to x} f_n(t)$. Then, we have.

① $\lim_{n \to \infty} A_n$ exists, say equal to A

② $\lim_{t \to x} f(t) = A$.

Rmk: if (t_m) is a seq in E, converge to x. Then.

we have a double seq $\{f_n(t_m)\}$.

$$\begin{array}{c|cccccc}
 & t_1 & t_2 & t_3 & \cdots & & \to x \\
\hline
f_1: & f_1(t_1) & f_1(t_2) & f_1(t_3) & \cdots & & \to A_1 \\
f_2 & f_2(t_1) & f_2(t_2) & \cdots & & & \to A_2, \\
f_3 & \vdots & \vdots & \vdots & & & \to A_3 \\
\vdots & \downarrow & \downarrow & \downarrow & & & \downarrow \\
\hline
f & f(t_1) & f(t_2) & f(t_3), & \cdots & \boxed{f(t_N)} & \to \lim_{t\to x} f(t) \;\cancel{=}\; \boxed{\lim A_n} \; ?
\end{array}$$

Pf: (1) (Lengthy version). $\forall \varepsilon>0$, need to find N, s.t.

$\forall n,m \geq N$, $\quad |A_n - A_m| < \varepsilon$.

We use unif Cauchy of $\{f_n\}$, to get a N, s.t.

$\forall n,m > N$.

$$\left|f_n(t) - f_m(t)\right| < \frac{\varepsilon}{3} \qquad \forall t \in E.$$

$\because$ $A_n = \lim_{t\to x} f_n(t)$, $\therefore \exists\ \delta_n > 0$. s.t.

if $0 < d_X(t,x) < \delta_n$, we have $|A_n - f_n(t)| < \frac{\varepsilon}{3}$.

$\therefore$ For any pair of $n,m > N$, take $t \in E$. s.t.

$0 < d(t,x) < \delta_n$ and $0 < d(t,x) < \delta_m$,

Then $$|A_n - A_m| \leq |A_n - f_n(t)| + |f_n(t) - f_m(t)| + |f_m(t) - A_m|$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.$$

(1) (sleek version). $\forall \varepsilon > 0$, need to find N, s.t.

$\forall n, m \geq N$, $\quad |A_n - A_m| \leq \varepsilon$.

We use unif Cauchy of $\{f_n\}$, to get a N, s.t.

$\forall n, m > N$.

$$(**) \quad \cdot \; |\underline{f_n(t) - f_m(t)}| \leq \varepsilon. \qquad \forall t \in E.$$

take limit $t \to x$ in $(**)$.

$$\lim_{t \to x} |f_n(t) - f_m(t)| = \left|\lim_{t\to x} f_n(t) - \lim_{t\to x} f_m(t)\right| = |A_n - A_m|.$$

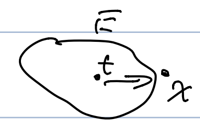
$$\therefore \quad |A_n - A_m| \leq \varepsilon \qquad \forall \; n, m > N.$$

Let $A = \lim_n A_n$

(2). Let $\varepsilon > 0$ be given, we want to show. $\exists \, \delta > 0$.

s.t. $\underline{\forall t \in X, \, 0 < d(t, x) < \delta}$, we have

$$|f(t) - A| \leq \varepsilon.$$



$\cdot \; |f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$. $\quad \forall n, \; \forall t \in X$.

$\cdot$ Using $f_n \to f$ uniformly, $\exists N_1$. s.t. $\forall n > N_1$,

$$|f_n(t) - f(t)| < \frac{\varepsilon}{3}.$$

$\cdot$ Using $A_n \to A$, $\quad \exists N_2$, s.t. $\forall n > N_2$.

$$|A_n - A| < \frac{\varepsilon}{3}.$$

Fix an n, s.t. $n > \max(N_1, N_2)$. Since $\lim_{t \to x} f_n(t) = A_n$

$\therefore \; \exists \, \delta_n > 0$, s.t. $\forall t \in E$, $0 < d(t, x) < \delta_n$, we have

$$|f_n(t) - A_n| < \frac{\varepsilon}{3}.$$

Hence, for this $\delta = \delta_n$, $\forall t \in X$, s.t. $0 < d(t, x) < \delta$,

we have $\quad |f(t) - A| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

i.e. $\lim_{t \to x} f(t) = A.$ #.

Thm: If $f_n : X \to \mathbb{R}$ cts fcn ;
if $f_n \to f$ uniformly
Then, f is continuous.

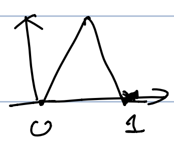
Pf: To show f is continuous, suffice to show
$\forall x \in X'$, we have $\lim_{t \to x} f(t) = f(x)$.

$$\lim_{t \to x} f(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t) \overset{\text{by unif conv.}}{=\!=} \lim_{n \to \infty} \underbrace{\lim_{t \to x} f_n(t)} \overset{\forall n.\ f_n \text{ is cont.}}{=\!=} \lim_{n \to \infty} f_n(x).$$

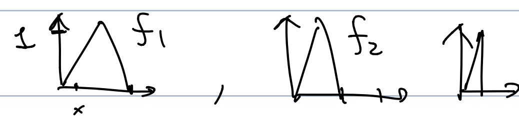
$$\underset{\because\ f_n(x) \to f(x)}{=} f(x)$$

#.

Q: If $f_n : K \to \mathbb{R}$ a seq of $\overbrace{\text{fcn}}^{\text{continuous}}$. on a compact set K.
If $f_n(x) \to f(x)$, pointwise, $\forall x \in K$.
Is it true that $f_n \to f$ uniformly ?

Ex: • $\varphi(x) =$  $= \begin{cases} \geq 0 & x \in [0,1] \\ 0 & \text{else where.} \end{cases}$

continuous. $\Rightarrow$. $\varphi(0) = 0$, $\varphi(1) = 0$

• $f_n(x) = \varphi(x \cdot n)$ 

$f_n(x) \to 0$ $\forall x \in [0,1] = K$.

But $\quad \overset{d_\infty(f_n,0)}{\overset{\shortparallel}{\sup_{x\in K} |f_n(x) - 0|}} = \sup_{x\in[0,1]} |\varphi(x)| = C > 0.$

not uniform convergence.

Ex: $\quad f_n(x) = x^n \qquad x \in [0,1]$.

Thm: Let K be a compact metric space. $f_n : K \to \mathbb{R}$.

Assume: (1) f_n is continuous, $\forall n$. ✓

(2). $f_n(x) \to f(x)$, $\forall x \in K$. and f is continuous. ✓

(3). $f_n(x) \geq f_{n+1}(x)$. $\forall x$, $\forall n$.

Then: $f_n \to f$ uniformly.

Pf: Let $g_n(x) = f_n(x) - f(x)$. Then $g_n \to 0$ pointwise on K.

and $g_n(x) \geq g_{n+1}(x)$. Suffice to show $g_n \to 0$ uniformly

$(\Rightarrow g_n(x) \geq 0)$

Let $\varepsilon > 0$ be given. we need to find $\underline{N > 0}$. s.t.

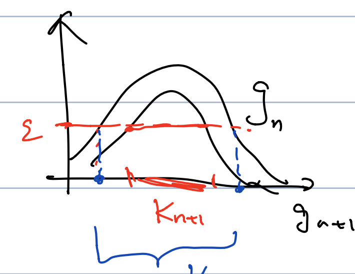
$\underline{\forall n \geq N, \quad g_n(x) < \varepsilon}$. $\forall n$, define

$K_n \subset K$, as $K_n = g_n^{-1}([\varepsilon, +\infty)) = \{x \in K \mid g_n(x) \geq \varepsilon\}$.

Then K_n is closed. ($\because g_n$ cont. and $[\varepsilon, +\infty)$ is closed)

~~and~~ K_n is compact ($\because$ closed subset of compact set is compact).

And $K_n \supset K_{n+1}$. Suffice to show that $\exists N$, s.t. $K_N = \emptyset$.



K_n.

Suppose $K_N \neq \phi$, $\forall N$. Then $\bigcap_{n=1}^{\infty} K_n \neq \phi$. (Thm 2.36)

Let $x \in \bigcap_{n=1}^{\infty} K_n$, then $g_n(x) \geq \varepsilon$ $\forall n$. This contradict with $g_n(x) \to 0$ $\forall x$. Hence $K_N = \phi$ for some N. ($\Rightarrow K_n = \phi$ $\forall n > N$). #