Sequence and Convergence of functions. (Ross Ch 4, Rudin Ch 7, section 1, 2).

· <u>Def</u>: Let $f: X \to Y$ be a map a <u>set X</u> to a <u>metric space Y</u>. Let $f_n: X \to Y$, for $n \in \mathbb{N}$, be a sequence of maps. We say f_n converges to f **pointwise**, $f_n \xrightarrow{\text{ptwise}} f$, if $\forall x \in X$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

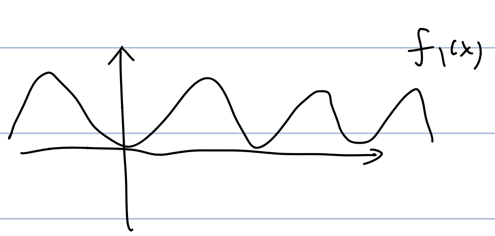
(convergences in the metric space Y)

For simplicity, consider $X = \mathbb{R}$, $Y = \mathbb{R}$.

<u>Ex</u>: (1) $f_n(x) = 1 + \frac{1}{n} \cdot \sin(x)$.

$\forall x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = 1$. So, let $f(x) = 1$ constant fcn.

we have $f_n(x) \to f(x)$, $\forall x \in \mathbb{R}$.



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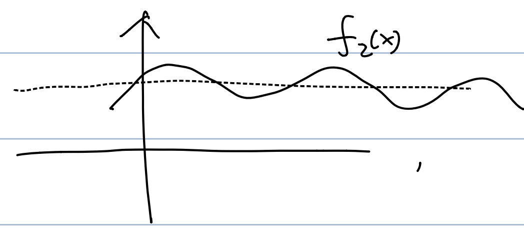

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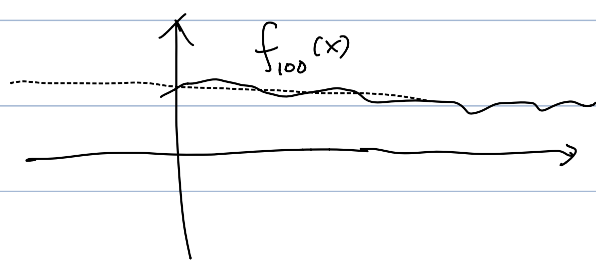

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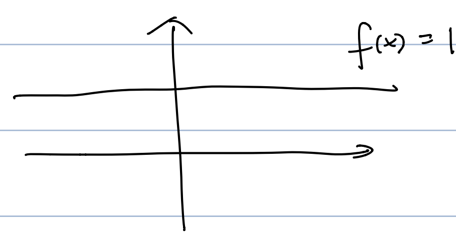
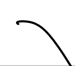


(2) "Bump function" supported on $[0, 1]$

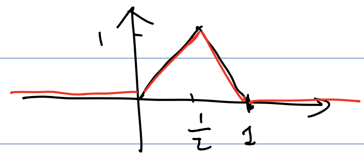


We say φ is a bump function ---- , if

$\varphi(x) = 0$ for $x \notin [0,1]$. $\varphi(x) \geq 0$ for $x \in [0,1]$.

$\varphi(x)$ is continuous.

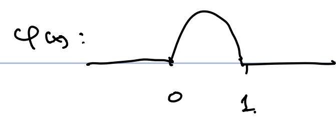
Ex:



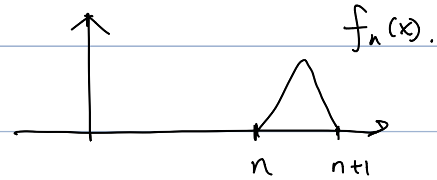
$$\varphi(x) = \begin{cases} 0 & x < 0 \\ 2x & x \in [0, \frac{1}{2}] \\ 1-2x & x \in [\frac{1}{2}, 1] \\ 0 & x > 1 \end{cases}$$

or. $$\varphi(x) = \begin{cases} 0 & x < 0 \\ x(1-x) & x \in [0,1] \\ 0 & x > 1 \end{cases}$$

$\varphi(x)$:



Let $f_n(x) = \varphi(x-n)$.



For any $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} f_n(x) = 0$

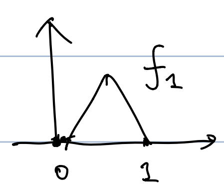
$\because$ $\forall x$, if $n > x$, then $f_n(x) = \underset{=0.}{\varphi(x-n)}$ has $x-n < 0$

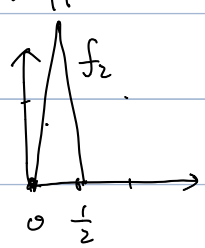
Hence, f_n pointwise converge to the function 0.

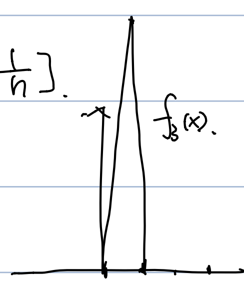
Ex 3 : (squeezed bump, stretched bump).

(a). $f_n(x) = n \cdot \varphi(nx)$.

$\Rightarrow$ f_n is supported in $[0, \frac{1}{n}]$.





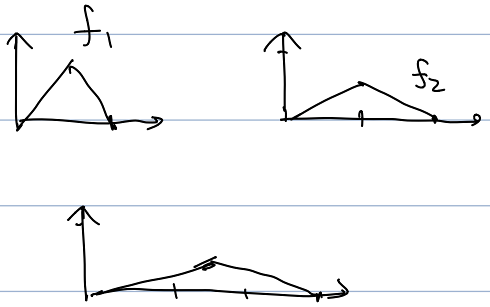


what's the pointwise limit? $\forall x \in \mathbb{R}$.

$$\lim_{n\to\infty} f_n(x) = 0$$

- $f_n(0) = 0. \Rightarrow \lim_{n\to\infty} f_n(0) = 0.$
- $\because \forall x \in (0,1)$, if n is large enough, $\frac{1}{n} < x$,

then $f_n(x) = 0$.



(b) $f_n(x) = \frac{1}{n} \varphi_n\left(\frac{x}{n}\right)$.

$\forall x \in \mathbb{R}$.

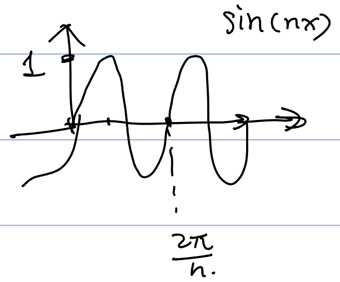
$$\lim_{n\to\infty} f_n(x) = 0$$

$\int_{\mathbb{R}} f_n(x)\,dx = \int \varphi(x)\,dx = C_0$ is constant, indep of n.

$\int_{\mathbb{R}} \lim f_n(x)\,dx = \int 0\,dx = 0$

so, pointwise limit of a function does not preserve integral.

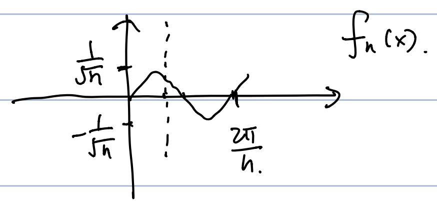
(4). $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$.



- $\lim_{n\to\infty} f_n(x) = 0.$

$\because$ $\left|f_n(x)\right| = \left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}}.$

$\lim_{n\to\infty} \left|f_n(x)\right| = 0.$



- $f_n'(x) = \frac{n \cdot \cos(nx)}{\sqrt{n}} = \underline{\underline{\sqrt{n}}} \cdot \cos(nx)$

Hence $f_n'(x)$ doesn't have a pointwise limit,

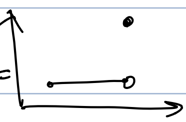
(for all $x \in \mathbb{R}$?) at least not at $x=0$.

$\because f_n'(0) = \sqrt{n}$.

If $f_n \to f$ ptwise, f_n' may not converge to f'.

(5) $f_n(x) = x^n$, $x \in [0,1]$.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1, & x = 1 \\ 0, & 0 \le x < 1 \end{cases}$$

the limit is discontinuous at $x=1$. $f(x)=$ 

- Review: sequence and convergence of "objects"

(a) object = real numbers $(x_n)_{n\in\mathbb{N}}$. $x_n \in \mathbb{R}$.

$x_n \to x$, if $\forall \varepsilon>0$, $\exists N>0$, s.t. $n>N$, $|x_n - x| < \varepsilon$.

(b). obj = points in $\mathbb{R}^d$.

A seq of ~~point~~ obj, $(\vec{x}_n)_{n\in\mathbb{N}}$.

$\vec{x}_n = (x_{n1}, x_{n2}, \cdots, x_{nd}) \in \mathbb{R}^d$.

2 possible notions of convergences:

(1). using $\mathbb{R}^d$'s Euclidean distance.

$$d_2(\vec{x},\vec{y}) = \left(\Sigma_{i=1}^{d} (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

Given a seq $(\vec{x}_n)$, we say $\vec{x}_n \to \vec{x}$, if

$$\lim_{n\to\infty} d_2(\vec{x}_n, \vec{x}) = 0.$$

(2). pointwise convergence.

View ~~an~~ an element $\vec{x} \in \mathbb{R}^d$, as

a map from $\underbrace{\{1,2,\cdots,d\}}^{=[d]} \to \mathbb{R}$

$$i \mapsto x_i$$

$$\mathbb{R}^d = \mathrm{Map}([d], \mathbb{R}).$$

We say $(\vec{x}_n)$ a seq of vectors in $\mathbb{R}^d$

converges to $\vec{x} \in \mathbb{R}^d$, pointwise (or. component-wise).

if, $\forall i \in [d]$, $x_{ni} \to x_i$.

$$\vec{x}_1 = (x_{11}, x_{12}, \cdots, x_{1d})$$

$$\vec{x}_2 = (x_{21}, x_{22}, \cdots, x_{2d}).$$

$$\vec{x}_3 = (\cdot\ \cdot \qquad)$$

$$\downarrow.$$

$$\vec{x} = (x_1, x_2, \cdots\cdots, x_d).$$

$\uparrow$ componentwise ~~conv~~ convergence.

<u>Lemma</u>: $(\vec{x}_n)$ converges to $\vec{x}$ in $\overset{d_2}{\checkmark}$ metric sense.

$\Leftrightarrow$ $(\vec{x}_n)$ converges to $\vec{x}$ componentwise.

$$\text{,, } \{1,\cdots,d\}.$$

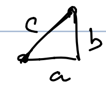
Pf: $\Rightarrow$ we need to show., $\forall i \in [d]$,

$x_{ni} \to x_i$, i.e. $\lim\limits_{n\to\infty} |x_{ni} - x_i| = 0$.

But we know $\lim\limits_{n\to\infty} \overbrace{\sqrt{|x_{n1} - x_1|^2 + \cdots + |x_{nd} - x_d|^2}}^{= d(\vec{x}_n, \vec{x})} = 0$.

and $d(\vec{x}_n, \vec{x}) \geq \sqrt{|x_{ni} - x_i|^2} = |x_{ni} - x_i|$

Hence $\lim\limits_{n\to\infty} |x_{ni} - x_i| = 0$



$(\sum a_i^2)^{\frac{1}{2}} \leq \sum |a_i|$. $c < a+b$.

$\Leftarrow$.

$\because 0 \leq d_2(\vec{x}_n, \vec{x}) \leq |x_{n1} - x_1| + |x_{n2} - x_2| + \cdots + |x_{nd} - x_d|$

and $\lim\limits_{n\to\infty} |x_{ni} - x_i| = 0$

$\therefore \lim\limits_{n\to\infty} (\text{RHS})_n = 0 \Rightarrow \lim\limits_{n\to\infty} d(\vec{x}_n, \vec{x}) = 0$

Q: if we replace $d_2(\vec{x}, \vec{y})$ by $d_\infty(\vec{x}, \vec{y})$.

$d_\infty(\vec{x}, \vec{y}) := \max\{ |x_i - y_i| : i \in [d] \}$,

is it still true that "convergence in d_∞ metric sense $\Leftrightarrow$ ~~pt~~ componentwise convergence"?

Ans: ✓. (more generally, if we have a norm function $\|\cdot\| : \mathbb{R}^d \to \mathbb{R}$.
and define $d(x, y) = \|x - y\|$.
then everything still holds.)

(c) obj = Map$(X, \mathbb{R})$. X : a set.

Here. take $X = [d]$, we get $\underline{\mathrm{Map}}\,(\underline{[d]}, \underline{\mathbb{R}}) = \mathbb{R}^d$.

From now on, take $\boxed{X = \mathbb{N}}$, Map $(\mathbb{N}, \mathbb{R})$ = the set of sequences.

$\forall n \in \mathbb{N}$,

Let $\vec{x}_n = (x_{n1}, x_{n2}, x_{n3}, \cdots\cdots) \in \mathrm{Map}(\mathbb{N}, \mathbb{R}) = \mathbb{R}^{\mathbb{N}}$

be a seq of ~~seq~~ real numbers,

Then, what do we mean, when we say,

$$\vec{x}_n \to \vec{x} \quad , \quad \text{for some } \vec{\alpha} \in \mathbb{R}^{\mathbb{N}}.$$

$$\vec{\alpha} = (\alpha_1, \alpha_2, \cdots\cdots)$$

- pointwise convergence :

  $\forall i \in \mathbb{N}$, require $x_{ni} \to x_i$ (as convergence in $\mathbb{R}$).

- d_2- metric sense convergence. , $\vec{x}_n \to \vec{x}$ in l^2 sense.

  if

  $$\lim_{n\to\infty} d_2(\vec{x}_n, \vec{x}) = 0, \quad \text{where}$$

  $$d_2(\vec{x}_n, \vec{x}) = \left(\sum_{i\in\mathbb{N}} |x_{ni} - x_i|^2\right)^{\frac{1}{2}}.$$

- d_∞ - metric sense convergence. $\vec{x}_n \to \vec{x}$ in l^∞-sense

  if $\lim_{n\to\infty} d_\infty(\vec{x}_n, \vec{x}) = 0$

  where.

$$d_\infty(\vec{x}_n, \vec{x}) = \sup\{ |x_{ni} - x_i| : i \in \mathbb{N} \}.$$

There 3 notions of convergence are all different.

Ex: $\vec{x}_n = (x_{n1}, x_{n2}, \cdots)$

$$x_{ni} = \frac{i}{n+i}$$

$$\begin{pmatrix} x_{11}, & x_{12}, & x_{13}, & \cdots \\ x_{21}, & x_{22}, & x_{23}, & \cdots \\ & & & \end{pmatrix} = \begin{pmatrix} \frac{1}{2}, & \frac{2}{3}, & \frac{3}{4}, & \frac{4}{5}, & \frac{5}{6}, & \cdots \\ \frac{1}{3}, & \frac{2}{4}, & \vdots & & & \\ \frac{1}{4}, & \frac{2}{5}, & & & & \\ \frac{1}{5}, & \frac{2}{6}, & & & & \\ \vdots & \vdots & & & & \end{pmatrix}$$

For fixed n,

$$\lim_{i\to\infty} x_{ni} = \lim_{i\to\infty} \frac{i}{n+i} = 1.$$

For fixed i, as $n \to \infty$, $\lim_{n\to\infty} \frac{i}{n+i} = 0$.

- Pointwise convergence, $\vec{x}_n \to \vec{0}$ i.e. $\forall i$ $\lim_{n\to\infty} x_{ni} = 0$

$(0, 0, \cdots 0)$

- d_2-convergence? $d_2(\vec{x}_n, \vec{0}) = \left(\sum_{i=1}^{\infty} |x_{ni} - 0|^2 \right)^{\frac{1}{2}}$

$$= \left(\sum_{i=1}^{\infty} x_{ni}^2 \right)^{\frac{1}{2}}$$

$\because \lim_{i\to\infty} x_{ni} = 1$, $\sum_{i=1}^{\infty} x_{ni}^2 = +\infty$,

Hence, $\vec{x}_n$ does not converge to 0 in the d_2-metric sense.

since $d_2(\vec{x}_n, \vec{x}) = +\infty$ $\forall n \in \mathbb{N}$, $\therefore \lim_{n\to\infty} d_2(\vec{x}_n, \vec{x}) \neq 0$.

$\vec{x} = \vec{0}$.

- d_∞ - sense ? $\quad d_\infty(\vec{x}_n, 0) = \sup\{|x_{ni} - 0| : i \in \mathbb{N}\}$

$$= \sup\left\{\frac{i}{n+i} : i \in \mathbb{N}\right\}.$$

$$= 1$$

The d_∞- distance between $\vec{x}_n$ and $\vec{0}$ stays constant, doesn't go to 0, Hence, $\vec{x}_n$ do not converge to. 0. in d_∞ sense.

- In our case, we will study:

  - obj = $\overset{\text{bounded}}{\text{continuous}}$ - Map $(\overset{X}{\overset{\shortparallel}{\mathbb{R}}}, \mathbb{R})$

  - we are going to consider pointwise convergence, & d_∞- metric sense convergence (a.k.a. uniform convergence.)