Recall: How to combine 2 convergent seqs to get new convergent sequence.

Thm: Given (a_n), (b_n). $\lim a_n = \alpha$, $\lim b_n = \beta$. Then

(1) • $\lim (a_n + b_n) = (\lim a_n) + (\lim b_n)$

(2) • $\lim (a_n \cdot b_n) = \lim (a_n) \cdot \lim (b_n)$

(3) • $\lim (a_n / b_n) = (\lim a_n) / (\lim b_n)$

if $b_n \neq 0$ $\forall n$, and if $\lim b_n \neq 0$.

Claim: $\lim (1/b_n) = 1/\lim b_n$.

Pf (3): $\forall \varepsilon > 0$. we need $N > 0$, s.t.

$$\left| \frac{1}{b_n} - \frac{1}{\beta} \right| < \varepsilon, \quad \forall n > N. \quad (*)$$

$$\Leftrightarrow \left| \frac{\beta - b_n}{b_n \cdot \beta} \right| < \varepsilon. \quad \forall n > N.$$

By the Lemma (last time), $\left\{ \frac{1}{b_n} \right\}$ is a bounded seq.

i.e. $\exists C > 0$, $\frac{1}{|b_n|} < C$ $\forall n$.

$$\because \left| \frac{\beta - b_n}{b_n \cdot \beta} \right| < \left| \frac{\beta - b_n}{\beta} \right| \cdot C.$$

$$\therefore \left| \frac{\beta - b_n}{b_n \cdot \beta} \right| < \varepsilon \Leftarrow \left| \frac{\beta - b_n}{\beta} \right| \cdot C < \varepsilon.$$

$\forall n > N$.

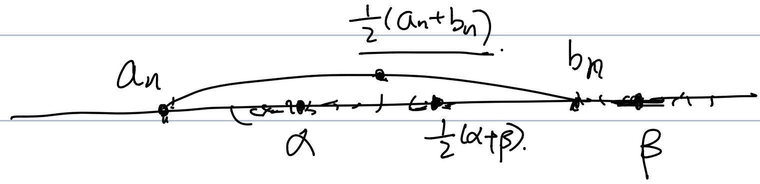
$$\Leftrightarrow |\beta - b_n| < \frac{\varepsilon}{C} \cdot |\beta|. \quad (**)$$

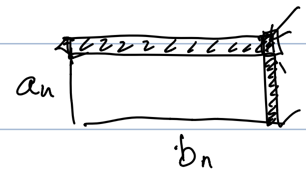
Now, $(**)$ can be satisfied by taking N large enough, thanks to $b_n \to \beta$. Hence $(*)$ is satisfied. # end of proof for the claim.

Finally, since $\frac{1}{b_n} \to \frac{1}{\beta}$, we have

$$\lim a_n \cdot \frac{1}{b_n} = \left(\lim a_n\right)\cdot\left(\lim \frac{1}{b_n}\right) = \alpha \cdot \frac{1}{\beta}. \qquad \#.$$

Intuitive Picture:

- 

- 

Ex:

- $$\lim \frac{n+3}{2n+5} = \lim \frac{1+\frac{3}{n}}{2+\frac{5}{n}} = \frac{1}{2}. \qquad \text{let } a_n = 1+\frac{3}{n},\ b_n = 2+\frac{5}{n}$$

- $$\lim \frac{n+3}{3n^2+5} = \lim \frac{1+\frac{3}{n}}{\left(3n+\frac{5}{n}\right)} = \lim \frac{\frac{1}{n}+\frac{3}{n^2}}{3+5/n^2} = \frac{0}{3} = 0.$$

$$a_n = \frac{1}{n}+\frac{3}{n^2}. \to 0$$

$$b_n = 3+5/n^2 \to 3$$

More Ex:

(1) $a_n = \frac{1}{n^p}$ $\qquad p>0.$

$\lim a_n = 0.$

if $p \in \mathbb{N}$. we can cheat: if we know $\lim \frac{1}{n} = 0$,

then $\lim \frac{1}{n^2} = \left(\lim \frac{1}{n}\right)\cdot\left(\lim \frac{1}{n}\right) = 0.0 = 0.$

$\vdots$

$$\lim \frac{1}{n^p} = \left(\underline{\lim \frac{1}{n^{p-1}}}\right)\cdot\left(\lim \frac{1}{n}\right) = 0\cdot 0 = 0.$$

To prove this directly: ~~let~~ $\forall \varepsilon>0$, need $N>0$. s.t.

$$\frac{1}{n^p} < \varepsilon \qquad \forall n>N.$$

$$\Leftrightarrow \frac{1}{\varepsilon} < n^p \qquad \forall n>N$$

$\Leftrightarrow \quad \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} < n, \qquad \forall\, n > N.$

$\therefore$ if we take $N = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$, then we are done.

(2). $a_n = n^{\frac{1}{n}}$,

claim: $\lim a_n = 1$.

$$\lim_{x\to\infty} x^{\left(\frac{1}{x}\right)} = \lim e^{\frac{1}{x}\log x} = e^0 = 1.$$

$$x = e^{\log x}$$

$$x^{\frac{1}{x}} = e^{\frac{1}{x}\cdot\log x} =$$

but $\frac{\log x}{x} \to 0$ as $x\to\infty$

$\Leftrightarrow \frac{u}{e^u} \to 0$ as $u\to\infty$

$u = \log x$.

Let $S_n = n^{\frac{1}{n}} - 1 > 0$ (if $n>1$)

suffice to show $\lim S_n = 0$.

$$1 + S_n = n^{\frac{1}{n}}$$

$$\Rightarrow (1+S_n)^n = n.$$

$$\Rightarrow 1 + n\cdot S_n + \frac{n(n-1)}{2!} S_n^2 + \cdots + S_n^n = n.$$

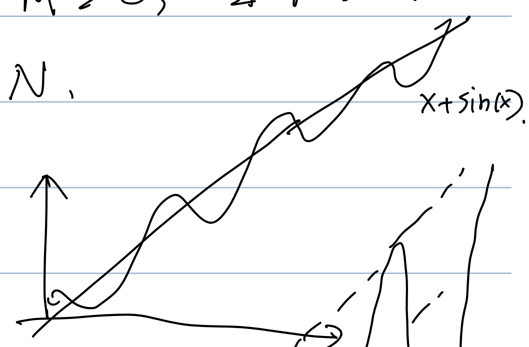
$$\Rightarrow \frac{\cancel{n}(n-1)}{2!} S_n^2 \le \cancel{n}.$$

$$\Rightarrow S_n \le \sqrt{\frac{2}{n-1}} \Rightarrow \lim_{n\to\infty} S_n = 0 \qquad \#$$

· Def: Let a_n be a seq of real numbers.

we say $\lim a_n = +\infty$, if $\forall M>0$, $\exists N>0$.

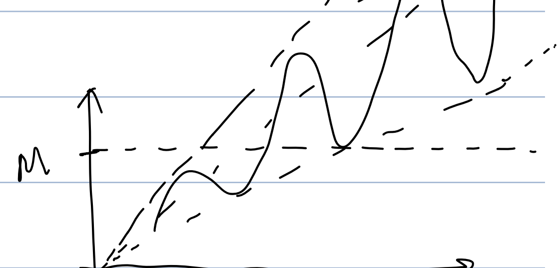
s.t. $a_n > M \quad \forall\, n > N$.



Ex: (1) $a_n = n + \sin(n)$

$\lim a_n = +\infty$ ✓.

(2). $a_n = n + \sqrt{n}\cdot\sin(n)$.



② $\lim a_n = +\infty$

$\because$ $a_n \geq n - \sqrt{n}$, but $\lim n - \sqrt{n} = +\infty$

$\therefore$ $\lim a_n = +\infty$

$\nearrow, \searrow$

§10 Monotone Seq and Cauchy Sequence.

Def: We say a seq (a_n) is increasing, if $a_{n+1} \geq a_n$

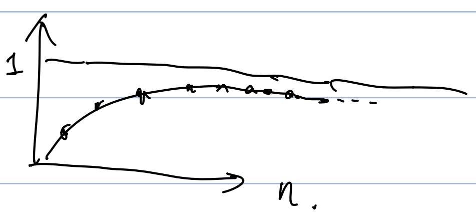
(equivally. if $a_m \geq a_n$ for all $m > n$)

(a_n) is decreasing, if $a_{n+1} \leq a_n$.

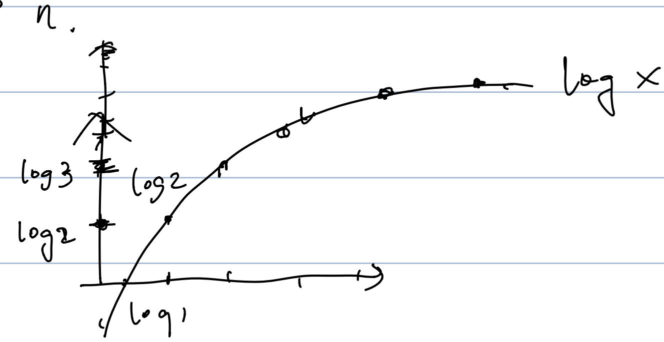
· Thm: If (a_n) is increasing and bounded, then (a_n) is convergent.

Ex: increasing seq: $a_n = n$, $a_n = n^2$,

$a_n = 1 - \frac{1}{n}$.



· $a_n = \log n$.



Pf: Let $S = \{a_n \mid n \in \mathbb{N}\}$. Since S is bounded, we have $\alpha = \sup(S)$. For any $\varepsilon > 0$, ~~the~~ $\alpha - \varepsilon$ is not an upper bound of S, i.e. $\exists\, n_0$, s.t. $a_{n_0} > \alpha - \varepsilon$. By monotonicity, $\forall\, n > n_0$, $a_n \geq a_{n_0} > \alpha - \varepsilon$. Thus,

$$\alpha - \varepsilon < a_n \leq \alpha \qquad \forall\, n \geq n_0.$$

$$\Rightarrow \qquad |a_n - \alpha| < \varepsilon \qquad \forall\, n \geq n_0$$

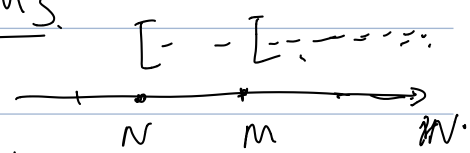
Hence $\lim a_n = \sup\{a_n\} = \alpha$. #,

- Thm: all bounded ~~bo~~ monotone seq are convergent.
- Def: lim sup, liminf.

- $S_N := \sup\{a_n \mid n \geq N\}$. (sup of the "tail" part of a_n).
- By construction:

  if $N < M$, then $S_N \geq S_M$.

  $\because \quad \underline{\{a_n \mid n \geq N\}} \supset \underline{\{a_n \mid n \geq M\}}$



If $A \supset B$ are subset of real number.

then $\sup A \geq \sup B$, $\inf A \leq \inf B$.

$\because$ if α is an upper bound for A, then α is also an upper bound for B, hence

$$\alpha \geq \sup B.$$

Now, take α to be the smallest upper bound for A, ~~hence~~. i.e. $\alpha = \sup A$. then

$$\sup A \geq \sup B.$$

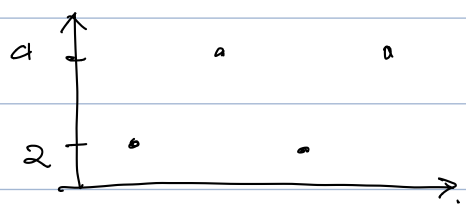
cf. $A \supset B$ finite set. $\max A \geq \max B$

$\min A \leq \min B$.

S_N is a decreasing sequence, hence by monotone convergence thm, (S_N) has a limit.

$$\limsup a_n := \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\sup_{n\geq N}(a_n)\right)$$

- Ex: •



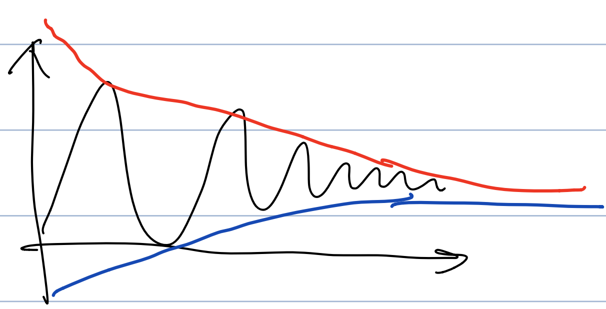
$a_n = 3 + (-1)^n$.

$a_n = 2, 4, 2, 4, \cdots$

- $\sup(a_n) = \limsup(a_n) = 4$.
- $\inf(a_n) = \liminf(a_n) = 2$.

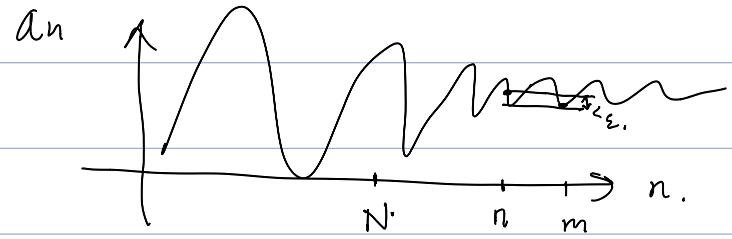
- $a_n = \underbrace{2, 0, 20, 1, 28}, 2, 4, 2, 4, \cdots$

$\limsup(a_n) = 4$, $\liminf(a_n) = 2$.



- Cauchy Sequence:

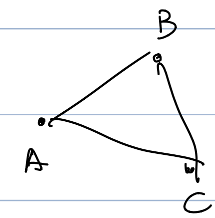
Def: (a_n) is a Cauchy seq. if $\forall \varepsilon > 0$, $\exists N > 0$.
s.t. $\forall n, m > N$, we have
$$|a_n - a_m| < \varepsilon.$$



Thm: (a_n) is a Cauchy seq $\Leftrightarrow$ (a_n) converges.

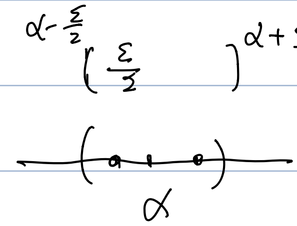
Pf: $\Leftarrow$ Assume a_n converges to $\underline{\underline{\alpha}}$. To show (a_n) Cauchy, we need to show $\forall \varepsilon > 0$, $\exists N > 0$. s.t.
$\forall n_1, n_2 > N$, $\quad |a_{n_1} - a_{n_2}| < \varepsilon$.

$$|a_{n_1} - a_{n_2}| = |(a_{n_1} - \alpha) - (a_{n_2} - \alpha)|$$
$$\leq |a_{n_1} - \alpha| + |a_{n_2} - \alpha|$$



dist(A, C)
$\leq$ dist(A, B)
+ dist(B, C).

So, if $\underline{N}$ is large enough, s.t. $\forall n > N$.



$$|a_n - \alpha| < \frac{\varepsilon}{2},$$

then $\quad |a_{n_1} - a_{n_2}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Such N exists, thanks to $a_n \to \alpha$.

$\Rightarrow$. **Lemma:** (a_n) converges, if and only if

$\limsup(a_n) = \liminf(a_n)$.

pf: $\Rightarrow$ Assume. $\lim a_n = \alpha$., we want to show $\underline{\limsup a_n}$ exists., and equals to α. $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n > N$, $|a_n - \alpha| < \varepsilon$.

in particular. $\underline{\alpha - \varepsilon} < \underline{a_n} < \alpha + \varepsilon \quad \forall n > N$.

$\alpha + \varepsilon$ is an upper bound for $\{a_n \mid n > N\}$.

$\therefore \quad \alpha + \varepsilon \geq S_N$. , $S_N > \alpha - \varepsilon$.

And by monotonicity of S_N, $S_N \geq S_{N+1} \geq \cdots$

$$|S_M - \alpha| < \varepsilon. \quad , \quad \forall M \geq N.$$

$\Rightarrow$. $S_N \to \alpha$.

Hence. $\limsup a_n = \lim a_n$.

similarly $\liminf a_n = \lim a_n$.

$\Leftarrow$ Suppose $\limsup a_n$ and $\liminf(a_n)$ exists, and equal. (next time).