Today: (1) finish §9.

(2) monotone seq. Cauchy seq.

Last time:

Thm: if (a_n), (b_n) seqs. $\lim a_n = \alpha$, $\lim b_n = \beta$. Then

(1). $\lim (a_n + b_n) = (\lim a_n) + (\lim b_n) = \alpha + \beta$

(2) $\lim a_n \cdot b_n = (\lim a_n) \cdot (\lim b_n) = \alpha \cdot \beta$.

☆ (3) if $b_n \neq 0$ $\forall n$, and if $\lim b_n \neq 0$.

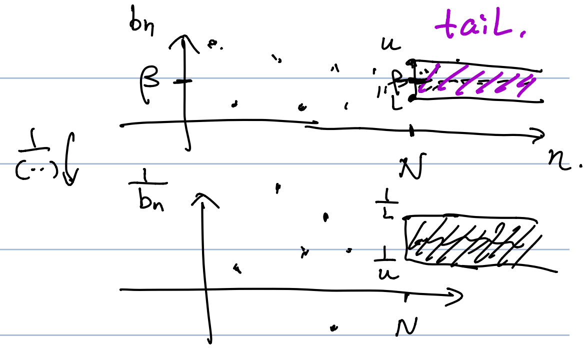
$$\lim (a_n / b_n) = \frac{\lim a_n}{\lim b_n}.$$

We finish proof of (3).

Lemma: If $b_n \neq 0$ $\forall n$, and if $\lim b_n = \beta \neq 0$.

then $\left(\frac{1}{b_n}\right)$ is bounded.

Pf: Without loss of generality (W.L.O.G) assume $\beta > 0$. (Otherwise, we can change b_n to $-b_n$, and β to $-\beta$).



(or any $\varepsilon > 0$, $\varepsilon < \beta$).

Fix $\varepsilon = \beta/2$. Then exist $N > 0$, s.t. $\forall n > N$, we have,

$$|b_n - \beta| < \varepsilon = \beta/2$$

$$\Leftrightarrow \beta - \frac{\beta}{2} \leq b_n \leq \beta + \frac{\beta}{2}$$

$$\Leftrightarrow \frac{\beta}{2} \leq b_n \leq \frac{3}{2}\beta.$$

(in addition, β, $b_n > 0$.) $\Rightarrow \frac{2}{\beta} \geq \frac{1}{b_n} \geq \frac{2}{3}\frac{1}{\beta}$. $\forall n > N$.

(actually, take $C_1 = 2/\beta$.)

Thus, there exists $C_1 > 0$, s.t. $\left|\frac{1}{b_n}\right| \leq C_1$, $\forall n > N$.

And. we take $C_2 = \max\{ |\frac{1}{b_n}| \mid 1 \leq n \leq N \}$.

$C = \max \{C_1, C_2\}$.

Thus, $\left|\frac{1}{b_n}\right| \leq C$ for all $n \in \mathbb{N}$. #

Finish (3) of theorem:

Pf: claim that $\lim \frac{1}{b_n} = \frac{1}{\beta}$. Indeed, $\forall \varepsilon > 0$,

we want to find $N > 0$. s.t.

$$\left|\frac{1}{b_n} - \frac{1}{\beta}\right| \lesssim \varepsilon \qquad \forall n > N. \qquad (*)$$

$$\Leftrightarrow \quad \left|\frac{\beta - b_n}{b_n \cdot \beta}\right| \lesssim \varepsilon \qquad \forall n > N.$$

$\because$ $\frac{1}{b_n}$ is bounded $\therefore \exists C > 0$. s.t. $\left|\frac{1}{b_n}\right| \leq C$.

$$\therefore \quad \left|\frac{\beta - b_n}{b_n \cdot \beta}\right| = \frac{|\beta - b_n|}{|\beta|} \cdot \frac{1}{|b_n|} \leq \frac{|\beta - b_n|}{|\beta|} \cdot C.$$

$$\Leftarrow \quad \left|\frac{\beta - b_n}{\beta}\right| \cdot C \leq \varepsilon.$$

$$\Leftrightarrow \quad |\beta - b_n| \leq \frac{\varepsilon}{C} \cdot |\beta|. \qquad (**)$$

since $b_n \to \beta$, we can find $N > 0$. s.t. $\forall n > N$.
$(**)$ is satisfied. Since $(**) \Rightarrow (*)$, we are done.

Finally
$$\lim a_n / b_n = \lim \left(a_n \cdot \frac{1}{b_n}\right) = \left(\lim a_n\right) \cdot \left(\lim \frac{1}{b_n}\right)$$
$$= \alpha / \beta.$$
#

Thm (of Examples)

(1). $\lim\limits_{n \to \infty} \frac{1}{n^p} = 0 \qquad \forall p > 0.$

(2). $\lim a^n = 0$ if $|a| < 1$.

(3) $\lim\limits_{n\to\infty} n^{1/n} = 1.$

(4) $\lim\limits_{n\to\infty} a^{1/n} = 1$ for $a>0$.

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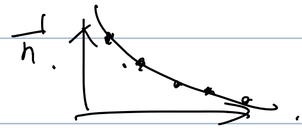
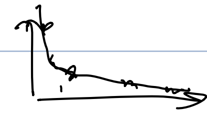
observation:

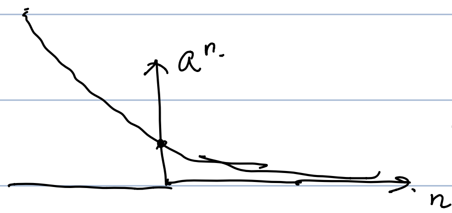
(1) $p=1$, $\frac{1}{n}$.



$p=2$, $\frac{1}{n^2}$.



(2). $1>a>0$,



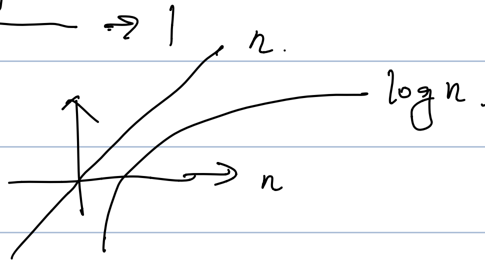
decrease to 0 faster than exponential decay.

$a^n = e^{-rn}$.

$a = e^{-r}$ $(r>0)$

$u=\log n$ is "weaker" than $n=e^u$.

(3). $n^{\frac{1}{n}} = e^{\frac{1}{n}\cdot\log n} \to 1$



$\frac{\log n}{n} \to 0.$

$e^0 = 1.$

(4). $a^{\frac{1}{n}} = e^{\overbrace{r\cdot\frac{1}{n}}^{\to 0}} \to e^0 = 1$

Pf: (1). $a_n = \frac{1}{n^p}$, $p>0$. $\forall \varepsilon>0$, we need $N>0$, s.t. $\forall n>N$.

$\left|\frac{1}{n^p} - 0\right| < \varepsilon. \iff \frac{1}{n^p} < \varepsilon.$

$\Leftrightarrow \quad \frac{1}{\varepsilon} < n^p. \quad \Leftrightarrow \quad \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} < n.$

$$\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}} = e^{\frac{1}{p}\log(\frac{1}{\varepsilon})} = \left[e^{\log(\frac{1}{\varepsilon})}\right]^{\frac{1}{p}}.$$

take $N = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}$ is enough.

(2). $a_n = a^n$. $\boxed{|a|<1}$

$\forall \varepsilon > 0$, need $N > 0$. s.t. $\forall n > N$.

$$|a^n - 0| < \varepsilon.$$

$$\Leftrightarrow \quad |a|^n < \varepsilon. \quad \Leftrightarrow \quad \left(\frac{1}{|a|}\right)^n > \frac{1}{\varepsilon}.$$

Since $|a| < 1$, we have $\frac{1}{|a|} = 1 + b$, for some $b > 0$.

thus. $\left(\frac{1}{|a|}\right)^n = (1+b)^n = \underbrace{1 + n\cdot b + \frac{n(n-1)}{2}\cdot b^2 + \cdots + b^n}_{1+n \text{ terms}} \geqslant nb.$

so $\left(\frac{1}{|a|}\right)^n > \frac{1}{\varepsilon} \quad \Leftarrow \quad nb > \frac{1}{\varepsilon}. \Leftrightarrow n > \frac{1}{\varepsilon \cdot b}$

take $N = \frac{1}{\varepsilon\cdot b}$ is enough.

(3). $a_n = n^{\frac{1}{n}}$. let $S_n = a_n - 1 = n^{\frac{1}{n}} - 1$. Then,

for $n \geqslant 1$, $S_n \geqslant 0$. Suffice to show $\lim S_n = 0$. We know

$$1 + S_n = n^{\frac{1}{n}}. \quad \Leftrightarrow \quad (1+S_n)^n = n.$$

$$n = 1 + n\cdot S_n + \frac{n(n-1)}{2}\cdot S_n^2 + \cdots\cdots$$

$$\geqslant \frac{n(n-1)}{2}\cdot S_n^2.$$

$$\therefore \quad S_n^2 \leqslant \frac{2}{n-1}$$

$$S_n \leqslant \sqrt{\frac{2}{n-1}}.$$

Thus $\lim S_n = 0.$ #.

(4). $\lim a^{\frac{1}{n}} = 1$ for $a > 0$.

If $a = 1$, then $1^{\frac{1}{n}} = 1$. constant seq.

If $a > 1$, then for $n > a$, we have

$$1 < a^{\frac{1}{n}} < n^{\frac{1}{n}}.$$

since $\lim n^{\frac{1}{n}} = 1$, we have

$$1 \leq \lim a^{\frac{1}{n}} \leq \lim n^{\frac{1}{n}} = 1.$$

$$\lim a^{\frac{1}{n}} = 1.$$

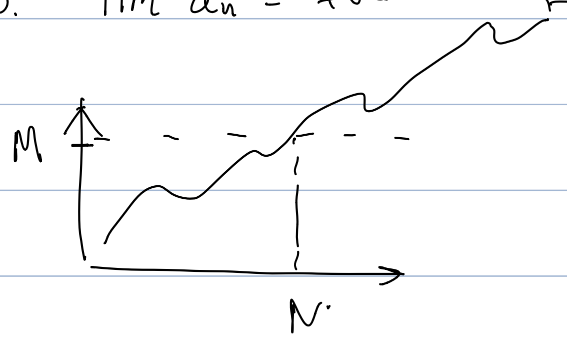
If $a < 1$, then $\lim a^{\frac{1}{n}} = \lim \frac{1}{\left(\frac{1}{a}\right)^{\frac{1}{n}}} = \frac{1}{\lim \left(\frac{1}{a}\right)^{\frac{1}{n}}} = \frac{1}{1}$.

$\frac{1}{a} > 1$ #.

Read about the rest in §9

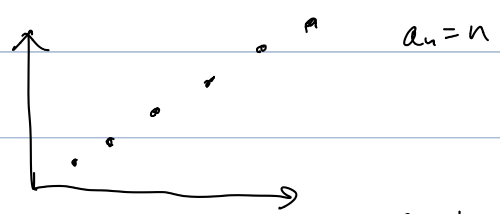
(1). $\lim a_n = +\infty \iff \forall M > 0, \exists N > 0$ s.t. $\forall n > N$,

$a_n > M$.

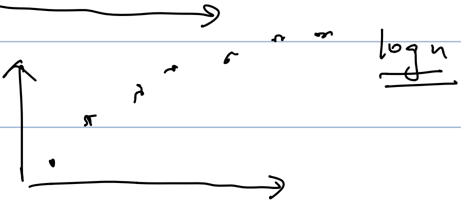


similarly $\lim a_n = -\infty \iff \lim (-a_n) = +\infty$.

§10 Monotone seq. and Cauchy Sequence.

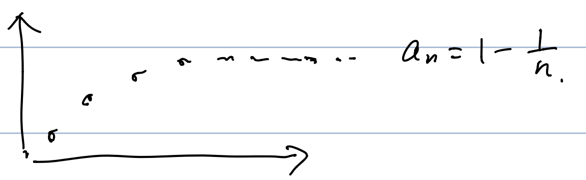


or



Def: (Cauchy Seq). (a_n) is a Cauchy seq, if $\forall \varepsilon > 0$, $\exists N > 0$. s.t. $\forall n_1, n_2 > N$, we have

bounded $\Rightarrow$
monotone
seq.



monotone increasing seq.

$|a_{n_1} - a_{n_2}| < \varepsilon$.

(oscillation amplitude gets smaller and smaller).

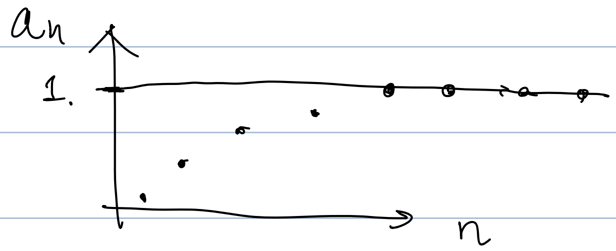
Def: • an increasing seq is such that $a_{n+1} \geq a_n$.

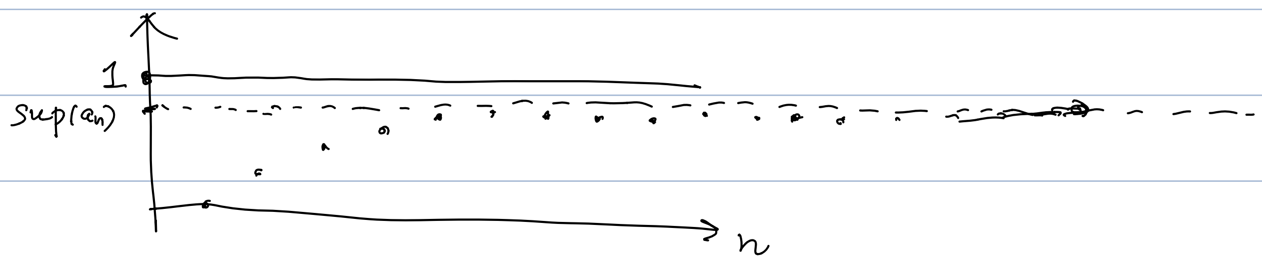
• a decreasing seq ———, $a_{n+1} \leq a_n$.

they are both called monotone seq.

Thm: All bounded Monotone seq are convergents.

Imagine a $\nearrow$ sequence (a_n), $a_n \leq 1 \quad \forall n$.





assume $a_n \nearrow$. $S = \{a_n \mid n \in \mathbb{N}\}$.

Pf: let $\alpha = \sup(a_n) = \sup S$, it exists by boundedness of a_n. We claim $\lim a_n = \alpha$.

$\forall \varepsilon > 0$, we know $\alpha - \varepsilon$ is not an upper bound of $\{a_n\}_{n\in\mathbb{N}}$. Hence, $\exists n_0$, s.t. $\alpha - \varepsilon < a_{n_0} \leq \alpha$.

By monotonicity, $\forall n > n_0$, $\alpha - \varepsilon < a_{n_0} < a_n \leq \alpha$.

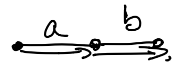
thus $|a_n - \alpha| < \varepsilon$. $\forall\, n > n_0$. #.

Cauchy sequence.

Thm: Let (a_n) be a sequence.

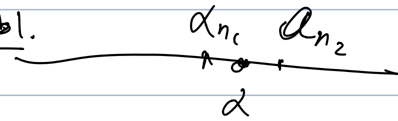
(a_n) is Cauchy $\iff$ (a_n) converge.

Pf: $\Leftarrow$ If $\lim a_n = \alpha$. To show (a_n) is Cauchy, we need to show $\forall\, \varepsilon > 0$, $\exists N > 0$. s.t. $\forall\, n_1, n_2 > N$, $|a_{n_1} - a_{n_2}| < \varepsilon$.

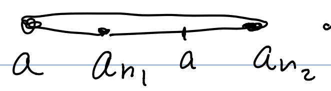


$a > 0,\ b > 0$

$|a+b| \le |a| + |b|$.



$\because$ $|a_{n_1} - a_{n_2}|$



$= |(a_{n_1} - \alpha) - (a_{n_2} - \alpha)|$

$\le |a_{n_1} - \alpha| + |a_{n_2} - \alpha|$ (triangle ineq).

if we take N large enough, s.t. $|a_n - \alpha| < \frac{\varepsilon}{2}$. $\forall\, n > N$, then we are done, i.e.

$|a_{n_1} - a_{n_2}| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Rightarrow$. (next time).