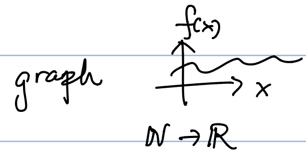
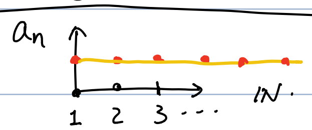
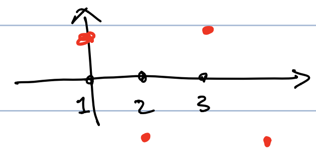
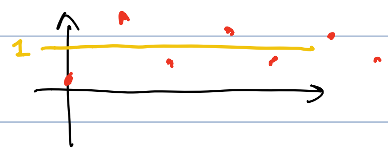
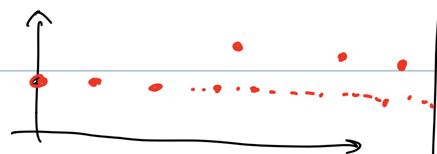
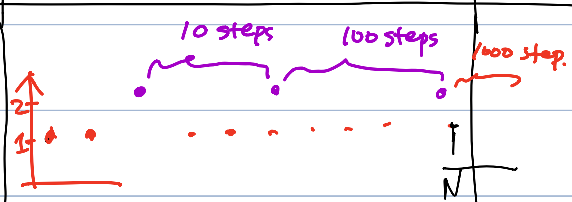
Math 104. Lec 3 (Ross §7, §9).

- Sequence: $a_1, a_2, a_3, \cdots$ $\quad a_i \in \mathbb{R} \quad i \in \mathbb{N}$.
- Limit: Let $a_1, a_2, \cdots$ be a sequence (of real numbers). We say a_i converge to $\alpha \in \mathbb{R}$ (real number), (denoted as $\lim_{n\to\infty} a_n = \alpha$), if $\forall$ ("for any") $\varepsilon > 0$, we can find $N > 0$, such that for any $n > N$, $n \in \mathbb{N}$, we have $|a_n - \alpha| < \varepsilon$.



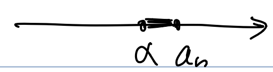
- Ex:

a_n	graph	limit.
1 constant seq.		$\lim a_n = 1$.
$a_n = (-1)^{n-1}$ $1, -1, 1, -1$		don't exist
$a_n = 1 + \frac{1}{n}(-1)^n$.		$\lim_{n\to\infty} a_n = 1$
$a_n = \begin{cases} 1 & n \not\equiv 0 \mod 10 \\ 1 + \frac{1}{2} & n \equiv 0 \mod 10 \end{cases}$ $n = 10, 20, 30, \ldots$		no limit.
		no limit.

$n \equiv k \mod m$ means $n = q \cdot m + k$

- To have a limit, $\forall \varepsilon > 0$ $|a_n - \alpha|$ needs to be $< \varepsilon$, for n large enough.

$|a_n - \alpha|$ = "distance between α a_n"



Ex: • $\lim_{n\to\infty} \frac{1}{n^2} = 0.$ $\qquad 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots$

Pf: $\forall \varepsilon > 0$, we need to find an N, such that (s.t.) $\forall n > N$.

$$|a_n - 0| < \varepsilon. \qquad (*)$$

This requirement $(*)$ $\Leftrightarrow$ $\frac{1}{n^2} < \varepsilon.$

$$\Leftrightarrow \frac{1}{\varepsilon} < n^2$$

$$\Leftrightarrow \frac{1}{\sqrt{\varepsilon}} < n.$$

So, if we take $N = 1/\sqrt{\varepsilon}$, then if $n > N$, we have $(*)$ satisfied.

• $\lim_{n\to\infty} \frac{2n+5}{3n+1} = \frac{2}{3}.$

idea: $2n+5 \approx 2n$, $\quad 3n+1 \approx 3n$.

$$\frac{2n}{3n} = \frac{2}{3}.$$

more rigorously. $\frac{2n+5}{3n+1} = \frac{2 + 5/n}{3 + 1/n} \to \frac{2}{3}$. (why we can take limit of top and bottom separately?)

Pf: $\forall \varepsilon > 0$, the requirement

$$|a_n - \tfrac{2}{3}| < \varepsilon$$

$$\Leftrightarrow \left| \frac{2n+5}{3n+1} - \frac{2}{3} \right| < \varepsilon.$$

$$\frac{2n+5}{3n+1} - \frac{2}{3} = \frac{(2n+5)3 - (3n+1)\cdot 2}{(3n+1)\cdot 3} = \frac{15-2}{(3n+1)\cdot 3} = \frac{13}{(3n+1)\cdot 3}$$

$$\Leftrightarrow \frac{13}{(3n+1)\cdot 3} < \varepsilon. \Leftrightarrow \frac{13}{3}\frac{1}{\varepsilon} < 3n+1.$$

$$\Leftrightarrow \frac{1}{3}\left(\frac{13}{3}\cdot\frac{1}{\varepsilon}-1\right) < n.$$

Let $N = \frac{1}{3}\left(\frac{13}{3}\frac{1}{\varepsilon}-1\right)$ (if this is negative, then set $N=0$).

then for any $n > N$, we have $(*)$ satisfied. #.

§9 Tools to compute limits.

<u>Slogan</u>: ① only the "tail" of a sequence matters.

② if a sequence <u>converge</u> to a limit, then the tail part of the seq is "under control".

<u>Thm</u>: All convergent sequences are bounded.

i.e. if $\lim\limits_{n\to\infty} a_n = \alpha$, then $\exists M > 0$, such that

$$-M \leq a_n \leq M \qquad \forall n \in \mathbb{N}. \qquad (\Leftrightarrow \ \underline{|a_n|} \leq M)$$

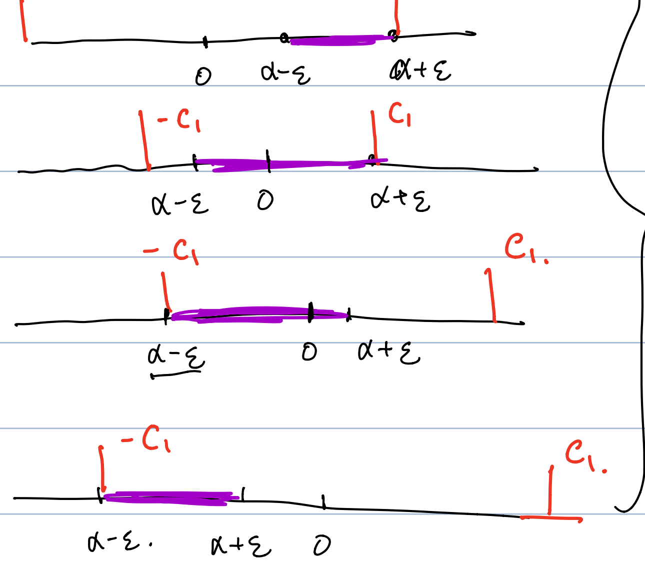
<u>Pf</u>: Fix an $\varepsilon > 0$, then $\exists N > 0$, s.t. $|a_n - \alpha| < \varepsilon$ $\forall n > N$.

Hence, $\alpha - \varepsilon < a_n < \alpha + \varepsilon$, $\forall n > N$.

let $C_1 = \max\{|\alpha-\varepsilon|, |\alpha+\varepsilon|\}$.

$C_2 = \max\{\underbrace{|a_n| \mid n \leq N \quad n \in \mathbb{N}}_{\text{a finite set.}}\}$

then. $\overset{(1)}{\forall}\ \overset{\mathbb{N}\ \Downarrow}{n > N}$, $|a_n| < C_1$



$\Rightarrow$ $(\alpha-\varepsilon, \alpha+\varepsilon) \subset (-C_1, C_1).$

$a_n \in (\alpha-\varepsilon, \alpha+\varepsilon)$

$\Rightarrow a_n \in (-C_1, C_1)$

$\Leftrightarrow |a_n| < C_1$

(2). $\forall n \leq N$, $|a_n| \leq C_2$ by construction.

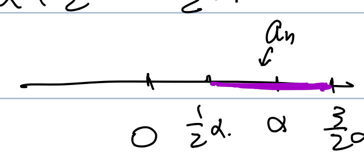
$\Rightarrow$ if we let $M = \max\{C_2, C_1\}$, then

$|a_n| \leq M$ for all $n \in \mathbb{N}$. #.

Prop: Suppose $\lim_{n\to\infty} a_n = \alpha$, and $a_n \neq 0$, $\forall n \in \mathbb{N}$, $\alpha \neq 0$. Then $\left(\frac{1}{a_n}\right)_{n\in\mathbb{N}}$ is a bounded sequence, i.e. $\exists M>0$, such that $\left|\frac{1}{a_n}\right| \leq M$ $\forall n \in \mathbb{N}$.

Rmk: • $a_n \neq 0$ $\forall n$ and $\alpha \neq 0$ do not imply each other.

Pf: Assume $\alpha > 0$ (otherwise, consider the seq $-a_n$.) Let $\varepsilon = \frac{1}{2}\alpha$, and let N be large enough, s.t. $\forall n > N$, $|a_n - \alpha| < \varepsilon$. $\Rightarrow \frac{1}{2}\alpha = \alpha - \varepsilon < a_n < \alpha + \varepsilon = \frac{3}{2}\alpha$



$$\Rightarrow \quad 0 < \frac{1}{\frac{3}{2}\alpha} < \frac{1}{a_n} < \frac{1}{\frac{1}{2}\alpha}$$

Let $C_1 = \max\left(\frac{1}{\frac{3}{2}\alpha}, \frac{1}{\frac{1}{2}\alpha}\right) = \frac{2}{\alpha}.$

Let $C_2 = \max\left\{\left|\frac{1}{a_n}\right| \;\middle|\; n \leq N\right\}.$

then $C = \max\{C_1, C_2\}$. Then $|1/a_n| \le C$ $\forall n \in \mathbb{N}$.

<u>Thm</u> Let $a_n \to \alpha$, $b_n \to \beta$. as $n \to \infty$.

(1) for any $k \in \mathbb{R}$.

$$\lim_{n\to\infty} (k \cdot a_n) = k \cdot \lim_{n\to\infty} (a_n) = k \cdot \alpha.$$

(2). $$\lim_{n\to\infty} (a_n + b_n) = (\lim a_n) + (\lim b_n)$$

(3) $$\lim (a_n \cdot b_n) = (\lim a_n) \cdot (\lim b_n)$$

(4) if $b_n \neq 0$ $\forall n$. and $\beta \neq 0$. then

$$\lim \left(\frac{a_n}{b_n}\right) = \frac{\lim a_n}{\lim b_n}.$$

if $k=0$, then nothing need to be shown. $\lim 0 = 0$. Assume $k \neq 0$,

<u>Pf</u>: (1) $\forall \varepsilon > 0$, the requirement

$$|k \cdot a_n - k\alpha| < \varepsilon$$

$$\Leftrightarrow \quad |a_n - \alpha| < \varepsilon / |k|.$$

Hence. $\exists N > 0$. s.t. $\forall n > N$. $|a_n - \alpha| < \varepsilon/|k|$.

(2). $\forall \varepsilon > 0$, the requirement

$$|(a_n + b_n) - (\alpha + \beta)| < \varepsilon \qquad (*)$$

$$\Leftrightarrow \quad |(a_n - \alpha) + (b_n - \beta)| < \varepsilon.$$

$$\Leftarrow \quad |a_n - \alpha| + |b_n - \beta| < \varepsilon.$$

$$\Leftarrow \quad |a_n - \alpha| < \frac{\varepsilon}{2}, \quad |b_n - \beta| < \frac{\varepsilon}{2}. \qquad (**)$$

we can find $N_1, N_2 > 0$, s.t. if $n > N_1$,

$|a_n - \alpha| < \frac{\varepsilon}{2}$, and if $n > N_2$, $|b_n - \beta| < \frac{\varepsilon}{2}$.

Thus. take $N = \max(N_1, N_2)$, then if $n > N$, then $(**)$ is satisfied, hence. $(*)$ is satisfied.

(3). $\forall \varepsilon > 0$, we need n be large enough, s.t.

$$|a_n b_n - \alpha \cdot \beta| < \varepsilon. \qquad (*).$$

$a_n = \alpha + (a_n - \alpha)$, $b_n = \beta + (b_n - \beta)$.

$$a_n \cdot b_n = [\alpha + (a_n - \alpha)][\beta + (b_n - \beta)]$$

$$= \alpha \cdot \beta + \underbrace{(a_n - \alpha)\beta + \alpha \cdot (b_n - \beta) + (a_n - \alpha)(b_n - \beta)}_{\text{fluctuation of the product } a_n b_n.}$$

For any $0 < \delta < 1$, $\exists N(\delta)$ s.t. $\forall n > N(\delta)$

$|a_n - \alpha| < \delta$, $|b_n - \beta| < \delta$.

Thus, $\forall n > N(\delta)$.

$$|a_n b_n - \alpha\beta| \leq |a_n - \alpha| \cdot |\beta| + |\alpha| \cdot |b_n - \beta| + |a_n - \alpha| \cdot |b_n - \beta|$$

$$\leq \delta \cdot |\beta| + |\alpha| \cdot \delta + \delta^2$$

$$\leq \delta(1 + |\alpha| + |\beta|).$$

. Hence, if we take $\delta_0 = \min\left(\frac{\varepsilon}{1 + |\alpha| + |\beta|}, 1\right)$. then.

$\delta_0(1 + |\alpha| + |\beta|) \leq \varepsilon$.

Thus, if we choose $N = N(\delta_0)$, then.

$|a_n b_n - \alpha\beta| \leq \delta(1 + |\alpha| + |\beta|) \leq \varepsilon.$ ✓.

(4). Let's first prove that $\lim \frac{1}{b_n} = \frac{1}{\beta}$.

$\forall \varepsilon > 0$, we need to find $N \geq 0$, s.t. $\forall n > N$, we have

$$\left| \frac{1}{b_n} - \frac{1}{\beta} \right| < \varepsilon. \qquad (*)$$

$$\Leftrightarrow \quad \left| \frac{\beta - b_n}{b_n \cdot \beta} \right| < \varepsilon.$$

$\because$ $\left|\frac{1}{b_n}\right|$ is bounded, $\therefore \exists C > 0$, s.t. $\left|\frac{1}{b_n}\right| < C$ $\forall n$.

thus.

$$\Leftarrow \quad \left| \frac{\beta - b_n}{\beta} \right| \cdot C < \varepsilon.$$

$$\Leftrightarrow \quad |\beta - b_n| < \frac{\varepsilon}{C} \cdot |\beta|. \qquad (**)$$

so, $\exists N > 0$, s.t. $\forall n > N$, $(**)$ is satisfied, hence $(*)$ is satisfied.

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