Today: ① $\mathbb{R}$, sup & inf, $\pm\infty$

② sequence $\{a_n\}$. limit.

Recall: ① $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$. Rational Roots of polynomials with $\mathbb{Z}$-coeff.

$\Rightarrow \sqrt{2} \notin \mathbb{Q}$

(1) Maximum and Minimum (of a set of numbers).

Def: Let S be a non-empty subset of real numbers.

(1) We say a real number α is a maximum of S, if $\alpha \in S$, and $\alpha \geq \beta$ $\forall \beta \in S$.

(2) — — — — — α is a minimum of S, if $\alpha \in S$, and $\alpha \leq \beta$, $\forall \beta \in S$.

Rmk: (1) if α_1, α_2 are both maximum of S, then

$\alpha_1 \geq \alpha_2$, $\alpha_2 \geq \alpha_1$ $\Rightarrow$ $\alpha_1 = \alpha_2$.

Thus maximum of S is unique (if it exists).,

we use $\max(S)$ to denote it.

(2). $\max(S)$ may not exists.

e.g. (for example). • $S = \mathbb{R}$,

• $S = (0,1)$ open interval.

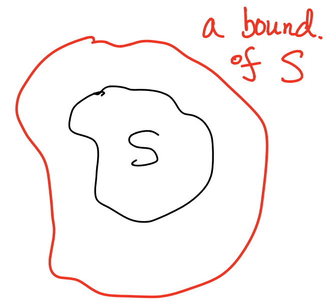
empty set.

(3). If $\emptyset \neq S \subset \mathbb{R}$ is a finite subset, then $\max(S)$ exist.

(2). Upper bound and Lower bound (of a set of numbers).

Ex: $S = (0,1)$. Then, any $\alpha \geq 1$, is an upperbound of S. Any $\alpha \leq 0$, is ~~an~~ a lower bound.

Same is true, for $S = [0,1]$, or $[0,1)$, $(0,1]$.



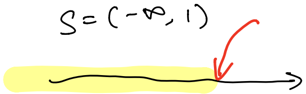
Def: Let $\emptyset \neq S \subset \mathbb{R}$.

- We say $\alpha \in \mathbb{R}$ is an upper bound of S, if
$$\alpha \geq \beta. \quad \forall \beta \in S$$
- $\alpha \in \mathbb{R}$ is a lower bound, if
$$\alpha \leq \beta \quad \forall \beta \in S.$$

Rmk: again, the upper or lower bound may not exists, (e.g. $S = \mathbb{R}$).

- If S has an upper bound, we say S is "bounded above".
- If S has a lower bound, we say S is bounded below.

$S = (-\infty, 1)$



bounded above.

$S = (-1, \infty)$

bounded below.

S is bounded.

If S is ~~a~~ bounded above,

- $\underbrace{\text{Least upper bound of } S}_{\sup(S)} := \min \{ \alpha \mid \underbrace{\alpha \text{ is an upper bound of } S}\}$.

sup short for supremum.

- $\underbrace{\text{if } S \text{ is bounded below, then}}$ great lower bound of $S := \max \{ \alpha \mid \alpha \text{ is a lower bound of } S\}$.

  " infimum of S "

  $= \inf(S)$.

Ex: (1). $S = \{1, 2, 3\}$.

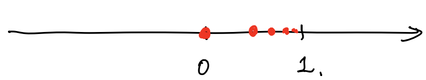
$\max(S) = 3$, $\sup(S) = 3$.

$\min(S) = 1$, $\inf(S) = 1$.

- if $\max(S) = \sup(S)$, $\inf(S) = \min(S)$, and if S is connected,
  $\Rightarrow$ S is a closed (bounded) interval.

- if $\max(S)$ exists, then $\sup(S) = \max(S)$.

(2). $S = \{ 1 - \frac{1}{n} \mid n \in \mathbb{N} \} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\}$



$\max(S)$ doesn't exist.

Pf: if it does exist, then it is of the form $1 - \frac{1}{n_0}$, for some $n_0 \in \mathbb{N}$. But

$S \ni 1 - \frac{1}{n_0+1} > 1 - \frac{1}{n_0}$.

this contradict with the requirement, that

$\max(S) \geqslant \beta \qquad \forall \beta \in S.$

$\sup(S) = 1.$

Pf: ① need to check 1 is an upper bounde. of S.

Indeed, $1 > 1 - \frac{1}{n}, \quad \forall n \in \mathbb{N}.$

② need to check that $\forall \alpha < 1$, α is not an upper bound. of S. (Not so rigourously), there exists an n, large enough, such that

$\alpha < 1 \Rightarrow \exists n \in \mathbb{N}, \quad \alpha < 1 - \frac{1}{n}. \quad \Leftarrow$ show $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$. s.t. $1/n < \varepsilon$.

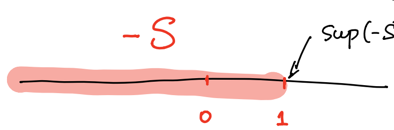
thus α is not an upper bounde of S.

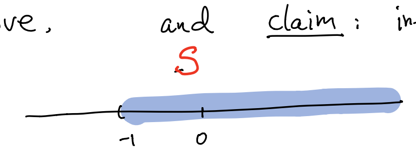
Completeness Axiom: Let $\emptyset \neq S \subset \mathbb{R}$.

If S is bounded from above, then $\sup(S)$ exists.

Cor: if S is bounded from below, then $\inf(S)$ exists.

Pf: Consider the set $-S = \{-x \mid x \in S\}$, then it is bounded from above, and claim: $\inf(S) = -\sup(-S)$.





Archimedean Property : (A.P.)

- if $a, b > 0$, then $\exists\, n \in \mathbb{N}$, s.t. $na > b$.
- <u>Pf</u>: Suppose A.P. fails for some pair of $a, b > 0$.

That is, $\forall\, n \in \mathbb{N}$, $na \leq b$. Let $S = \{ na \mid n \in \mathbb{N} \}$.

then by assumption on a, b, S is bounded above by b.

By <u>completeness axiom</u>, $\sup(S)$ exists. denoted as S_0.

$$S_0 > S_0 - a \qquad (\because a > 0).$$

that means, $S_0 - a$ is not an upper bound of S.

(since S_0 is the minimum of all possible upper bound).

$\Rightarrow$ $\exists\, n \in \mathbb{N}$, s.t. $na > S_0 - a$.

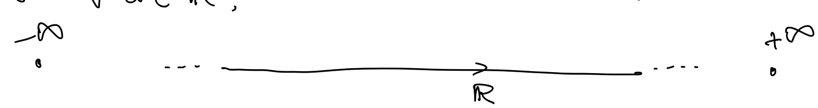
$\Rightarrow$ $(n+1)a > S_0$

$\Rightarrow$ S_0 is not an upper bound of S.

contradiction! #.

- <u>$+\infty$, $-\infty$.</u>
  - $\hat{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ as a set with ordering.

s.t. $\forall\, a \in \mathbb{R}$, $-\infty < a$, and $a < +\infty$



- If $\emptyset \neq S \subset \mathbb{R}$, we say $\sup(S) = +\infty \Leftrightarrow S$ is not bounded from above.

similarly, $\inf(S) = -\infty$ means. S is not bounded below.

Ch 2. Sequences & Limits.

(§7). Example & Definitions.

A sequence of real numbers, is the following data.

$a_1, a_2, a_3, \cdots\cdots$ $\quad a_n \in \mathbb{R} \quad \forall n \in \mathbb{N}$.

[more formally, a function, $\mathbb{N} \to \mathbb{R}$.
$n \mapsto a_n$.]

Ex: (1) constant sequence:

$3, 3, 3, \cdots$

(2). $1, 2, 3, 4, \cdots\cdots$

(3). $1, -2, 3, -4, \cdots\cdots$

(4). $1, \frac{1}{2}, \frac{1}{3}, \cdots$ $\quad a_n = \frac{1}{n}$.

(5). $1, 2, 4, 8, \cdots$ $\quad a_n = 2^{n-1}$

Ex: How to construct a sequence of rational numbers that gets closer and closer to $\sqrt{2}$?

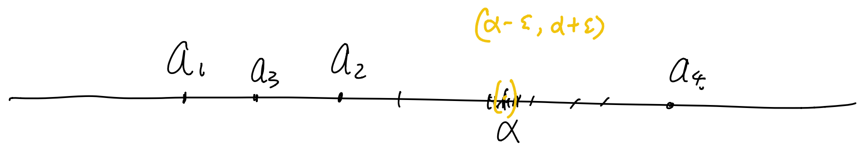
one way: write $\sqrt{2}$ as decimal $1.414\cdots$

then define $a_n = \underbrace{1.414\cdots}_{\text{keep } n \text{ digits after the period.}} = 1 + \frac{414\cdots}{10^n} \in \mathbb{Q}$

Rmk: Sequence is useful for "approximation".

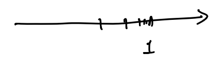
Definition (Limit): We say a sequence $(a_n)_{n\in\mathbb{N}}$, has limit $\alpha \in \mathbb{R}$, if $\forall \varepsilon > 0$. $\exists N > 0$. such that for all positive integer n, with $n \geq N$, we have.

$|a_n - \alpha| < \varepsilon$. $\lim_{n\to\infty} a_n = \alpha$.

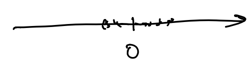


claim:

Ex: (1) $a_n = 1 - \frac{1}{n}$. , $\lim a_n = 1$.



(2). $a_n = \frac{1}{n} \cdot \sin(n)$.



$\lim_{n} a_n = 0$.

non-trivial.

(3). $a_n = \left(1+\frac{1}{n}\right)^n$ $\lim_{n\to\infty} a_n = e$.