Today's Plan:

1. (§4). Completeness Axiom.
2. (§7). Definition of Seq & Limit.

Recall last time:

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$,
- Rational zeros of polynomials. $\Rightarrow$ $\sqrt{2}$ is not a rational number.

Today:

Def'n $\qquad S \neq \emptyset$

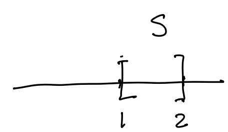
- max, min: Let $S \subset \mathbb{R}$, we say an element $\alpha \in S$ is a maximum, if $\forall \beta \in S$, $\alpha \geq \beta$. Similarly, $\alpha \in S$ is a minimum, if $\forall \beta \in S$, $\alpha \leq \beta$.

Rmk:
- max, min are inside the set S.
- they are not guaranteed to exist.

Ex: (1). $S = \{1, 2, 3, 4\}$. $\max(S)$ exists, $\max(S) = 4$

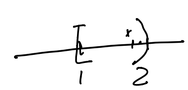
$\min(S)$ exists, $\min(S) = 1$.

(2). $S = [1, 2] = \{x \mid 1 \leq x \leq 2\}$

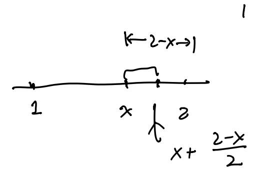


$\max(S) = 2$, $\min(S) = 1$.

(3) $S = [1, 2) = \{x \mid 1 \leq x < 2\}$



$\max(S)$ does not exist.



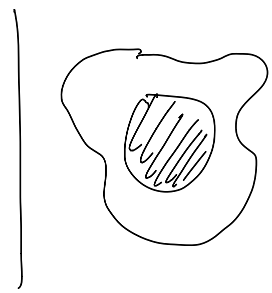
$\forall x \in S$, $\quad x + \left(\frac{2-x}{2}\right) \in S$,

$\qquad x + \left(\frac{2-x}{2}\right) > x$.

$\Rightarrow$ x is not a maximum.

Q1: why not use $1.999\cdots$?

it turns out, $1.999\cdots = 2 \notin S$.

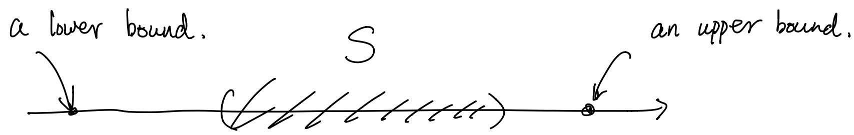


- Upper bound, lower bound.

Def'n Let $\emptyset \neq S \subset \mathbb{R}$. (empty set)

(a) Let $\alpha \in \mathbb{R}$. We say α is an upper bound of S. if $\forall \beta \in S$, $\beta \leq \alpha$.

(b) Let $\alpha \in \mathbb{R}$, α is an lower bound of S, if $\forall \beta \in S$, $\alpha \leq \beta$.

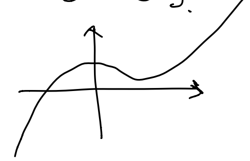


Rmk: • they are far from unique.

• may not exists. (e.g. $S = \mathbb{R}$).

Ex: • $S = \{$ roots in $\mathbb{R}$ of $x^3 - x + 5 = 0\}$.

(Q: is S non empty? yes. graph of $x^3 - x + 5$)



100 is an upperbound of S.

$\Leftrightarrow$ $x \geq 100$, then $x^3 - x + 5 > 0$.

hence $x \notin S$. Thus, if $x \in S$, $x < 100$.

- $S = [5, 10]$, an upper bound can be taken as 11. (or $S = (5, 10)$, $S = [5, 10)$.)

1 is a lower bound.

- 10 is an upperbound, 5 is a lower bound.

- **Def:** (sup, inf). Let S be a non-empty subset of $\mathbb{R}$. ($\emptyset \neq S \subset \mathbb{R}$).

  (a) if S is bounded above (i.e. there exists an upper bound for S), and S has a least upper bound, then we call it the supremum of, denoted as $\sup S$.

  (b). if S is bounded below and S has a greatest lower bound, then we call it infimum of S, denoted as $\inf S$.

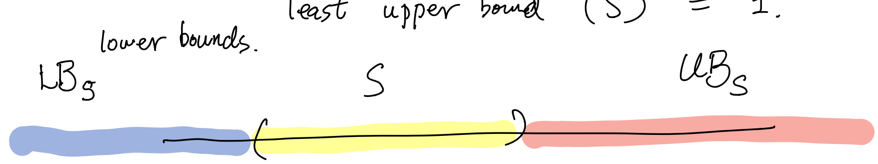
- $UB_S = \{x \mid x \text{ is an upper bound of } S\}$.

  S is bounded above $\Leftrightarrow$ UB_S is nonempty

  S has a least upper bound $\Leftrightarrow$ $\min(UB_S)$ exists.

Ex: • $S = (0, 1)$, $\max(S)$ doesn't exist.

least upper bound $(S) = 1$.



- $S = [0, 1]$, $\max(S) = 1$.

  $\max(S)$, if exists, is always an upper bound.

  actually, the least upper bound.

- <u>Completeness Axiom</u>:

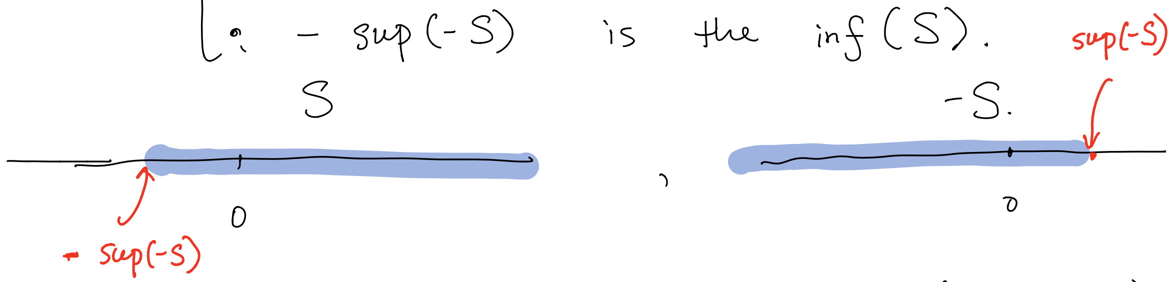
Every non-empty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound.

- <u>Cor</u>: if S is bounded from below, then $\inf S$ exists.

~~Pf~~ idea

~~Pf~~: consider $-S = \{-x \mid x \in S\}$.

Then $\begin{cases} \bullet\ -S \text{ is bounded from above.} \\ \bullet\ -\sup(-S) \text{ is the } \inf(S). \end{cases}$



(read the actual proof in Ross).

Pf: Since S is bounded from below, hence $\exists\, m \in \mathbb{R}$, s.t. $m \le s$, $\forall s \in S$. $\Rightarrow$ $-m \geq -s$, $\forall s \in S$.

i.e. $-m \geq u$. $u \in -S$.

hence $-m$ is an upper bound of $-S$.

Now, apply the completeness Axiom, $\sup(-S)$ exists.

let $\alpha = \sup(-S)$. Claim. $\inf(S) = -\alpha$. We need to prove $\begin{cases} \bullet\ -\alpha \text{ is a lower bound of } S \\ \bullet\ -\alpha \text{ is the greatest lower bound of } S. \end{cases}$

$\Leftrightarrow$ $\begin{cases} \bullet\ \alpha \text{ is an upper bound of } -S. \\ \bullet\ \alpha \text{ is the least upper bound of } -S. \end{cases}$

#.

<u>Archimedian Property</u>:

If $a>0$, $b>0$ are real numbers, then for some $n \in \mathbb{N}$, we have $n \cdot a > b$. ($\Leftrightarrow$ $a > \frac{b}{n}$)

<u>Pf</u>: We <u>assume</u> Archimedian property does not always hold. That is, $\exists$ $a>0, b>0$, such that for all $n \in \mathbb{N}$. $n \cdot a \leq b$.
Let $S = \{ na \mid n \in \mathbb{N} \}$. Then b is an upper bound of S. By completeness Axioms, we have $\sup(S)$ exists. Let $S_0 = \sup(S)$. Since $a>0$, we have

$$S_0 < S_0 + a.$$

$$\Rightarrow \quad S_0 - a < S_0.$$

By the definition of least upper bound, $S_0 - a$ is not an upper bound of S. Then $\exists$ $na \in S$, s.t.

$$na > S_0 - a. \quad \Rightarrow \quad na + a > S_0.$$

But $na + a = (n+1) \cdot a \in S$, so $na + a > S_0$ <u>contradict</u> with S_0 being an upper bound of S. Hence the assumption that A.P. fails is wrong.

<u>Rmk</u>: • $\forall$ $a>0$, no matter how small, $\exists$ $n \in \mathbb{N}$.

$$a \cdot n > 1, \quad \Leftrightarrow \quad a > \frac{1}{n}.$$

· <u>§7</u>. Definition of Sequence and Limits of real numbers.

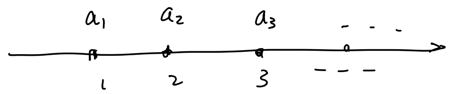
- A sequence is a collection of ~~ordered~~ numbers, indexed by $n \in \mathbb{N}$.

$a_1, a_2, a_3, \cdots$ where $a_n \in \mathbb{R}$.

(a function $\mathbb{N} \to \mathbb{R}$).

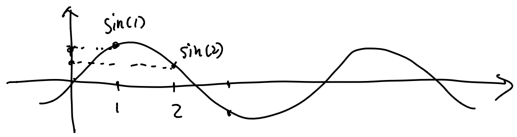
Ex: • $a_1 = 1, \quad a_2 = 2, \quad a_3 = 3, \cdots, \quad a_n = n,$

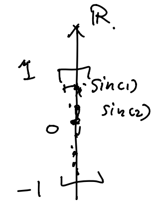
(no limit).



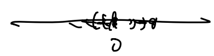
$\left\{\begin{array}{l}\text{limit exist}\\ \text{equal} = 1\end{array}\right)$ • $a_n = 1, \quad \forall n \in \mathbb{N}$. constant sequence.

(no limit) • $a_n = \sin(n)$.





(limit = 0) • $a_n = \frac{1}{n} \cdot \sin(n)$.



(limit = 0) • $a_n = \frac{1}{n} \cdot (-1)^n$. $\quad -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \cdots\cdots$

H. a dot jumping on the real line, forever, non-stop !!

• Read : $\left(\begin{array}{ll}\S 4. & \text{Density of } \mathbb{Q} \text{ in } \mathbb{R}.\\ \S 5. & -\infty, +\infty.\end{array}\right)$.