Plan for this semester: (Ch2) Ross.

- $\mathbb{R}$, convergence of sequence of numbers. $\{a_n\}$. $\begin{pmatrix}\text{midterm}\\ 1\end{pmatrix}$
- function, convergence of sequence of functions. (midterm 2)
- $\frac{d}{dx}$, $\int$.

- Today: $\mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z} \supset \mathbb{N}$.
  - $\mathbb{N}$: natural number. $\{1, 2, 3, 4, \cdots\}$.

    allowed operation: $+$, $\cdot$,

    - "successor" : e.g. "the successor of 3 is 4",
- Property of $\mathbb{N}$; if $S \subset \mathbb{N}$, such that
  - $1 \in S$
  - $n \in S \Rightarrow n+1 \in S$

  then $S = \mathbb{N}$.

- Mathematical induction:
  - let $P_n$ be a proposition, that depends on $n \in \mathbb{N}$.

    e.g. $P_n$ = "$n(n+1)$ is an even number"
  - induction is a way to prove $P_n$ holds for all $n \in \mathbb{N}$.

  by
  - show $P_n$ is true for $n=1$
  - show that if $P_n$ is true, then $P_{n+1}$ is true.

if we let $S = \{n \mid P_n \text{ is true}\}$, then $S = \mathbb{N}$.

$\Rightarrow$ $P_n$ is true for all $n \in \mathbb{N}$.

- Ex: (from Ross)

  (1) $|\sin(nx)| \leq n \cdot |\sin(x)| \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}$.

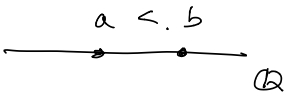
  (2) see the book.

- $\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \cdots\}$.
  - $+, \cdot, -$ ↙ subtraction.
  - 0 is distinguished, in that $0 + a = a$.
    1 is — ——, in that $1 \cdot a = a$.
  - $\forall n \in \mathbb{Z}$, $\exists$ a unique element in $\mathbb{Z}$, i.e. $-n$. $(-n) + (n) = 0$. ↙ (c.f. Abstract Algebra).

$\mathbb{Z}$ is an example of a "ring", i.e. a gadget that support $+, \cdot, -$. has the analog of $0, 1, \cdots$.

Other example of ring: $\mathbb{Z}[x]$, integer coefficient polynomial

- $\mathbb{Q}$. rational number, $\{\frac{n}{m} \mid n, m \in \mathbb{Z}, m \neq 0\}$.
  - operation it supports: $+, -, \cdot, /$.
    division.
  - $\mathbb{Q}$ is an example of a "field"
    see definition in Ross.
  - $\mathbb{Q}$ is an order field, i.e. there is a notion of "$\leq$". (e.g. $3 \leq 4$).
    see definition in Ross.

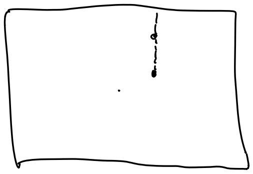


claim:
$\mathbb{C}$ is not an ordered field.
$= \{a + ib \mid a, b \in \mathbb{R}\}$.

Q1: why not. Can I declare that

$$(a,b) \leq (c,d) \quad \text{mean} \quad \begin{cases} a < c \\ \text{or if } a = c,\ b < d \\ \text{or } (a,b) = (c,d). \end{cases}$$

$\mathbb{C} \simeq \mathbb{R}^2$



- $\mathbb{Q}$ is not all the story. i.e.

solve equation $x^2 - 2 = 0$,

claim: $\underline{\pm\sqrt{2}}$ are not in $\mathbb{Q}$.

(not every expression involving $\sqrt{\ }$ is irrational, see Ex. 2.7 in Ross).

Prop: If $r = \frac{\overset{\neq 0}{c}}{d} \in \mathbb{Q}$ is a rational number, and $r$ satisfies the equation

$$C_n \cdot x^n + C_{n-1} \cdot x^{n-1} + \cdots + C_0 = 0$$

with $C_i \in \mathbb{Z}$, $C_n \neq 0$, $C_0 \neq 0$.

then. $\begin{cases} d \mid C_n \\ c \mid C_0. \end{cases}$ ✓

Pf: Plug in $r = \frac{c}{d}$, $(\gcd(c,d) = 1)$, to the eq.

$$C_n \left(\frac{c}{d}\right)^n + \cdots + C_0 (1) = 0$$

$\cdot d^n \downarrow$

$$C_n \cdot c^n + C_{n-1} \cdot c^{n-1} \cdot d + \cdots + C_1 \cdot c^1 \cdot d^{n-1} + \underline{C_0 \cdot d^n} = 0.$$

(1). $$C_0 \cdot d^n = -\left(C_n \cdot c^n + C_{n-1} \cdot c^{n-1} \cdot d + \cdots + C_1 \cdot c \cdot d^{n-1}\right).$$

$$= -c \cdot \big(\underbrace{C_n \cdot c^{n-1} + C_{n-1} \cdot c^{n-2} \cdot d + \cdots + C_1 \cdot d^{n-1}}_{\text{an integer,}}\big).$$

$\Rightarrow$ $c \mid C_0 \cdot d^n$,

more over, since $c$ and $d$ are co-prime,

$\Rightarrow$ $c \mid C_0$.

(2). similarly, if we single out the first term.

$c_n \cdot c^n = -d \cdot ($ some integer$)$.

$\Rightarrow \quad d \mid c_n \cdot c^n \quad \underset{\nearrow}{\Rightarrow} \quad d \mid c_n$. #

∵ $c, d$ coprime.

Cor: if $\underset{\text{leading coeff.}}{c_n = 1}$. then $d = 1$, i.e. $r$ is an integer.

Pf of claim that "$\sqrt{2}$ is not in $\mathbb{Q}$":

suppose $\sqrt{2}$ is rational, then since $\sqrt{2}$ is root of $x^2 - 2 = 0$. then $\sqrt{2}$ is an integer, which is impossible., hence $\sqrt{2}$ is not rational. #.

- What is in $\mathbb{R}$ but not in $\mathbb{Q}$?
  - fact: it is not enough to add in just real roots of polynomial equations with $\mathbb{Z}$- coefficient.
  - i.e. $\exists\, b \in \mathbb{R}$, s.t. $b$ does not satisfy any polynomial eq. with $\mathbb{Z}$- coeff.

    For example: $\pi$, $e$. are such number. "transcendental number".

- $\mathbb{R} \supset \mathbb{Q}$ and satisfies a completeness axiom:

i.e. if $S \subset \mathbb{R}$ is bounded from above, then

$\sup(S)$ exists in $\mathbb{R}$.

least upper bound.