Today: (1) HW1

(2). example of recursive seq

(3). §11 subsequence.

(1) HW $\sqrt{2+\sqrt{2}} \notin \mathbb{Q}$ $\Longleftrightarrow$ $\underset{\Updownarrow}{\overset{\alpha}{\sqrt{2+\sqrt{2}}}}$ is not $\pm 1, \pm 2$.

$\pm 1, \pm 2$ is not a root of $x^4 - 4x + 2 = 0$.

(2). Ex 2 on P58 of Ross.

- goal is to use monotone boundeness of the seq, to show that its limit exists (in $\mathbb{R}$).

- $S_1 = 5$, $S_{n+1} = \dfrac{S_n^2 + 5}{2 S_n}$.

hypothesis: $\begin{cases} A_n: & \forall.\ S_n \leq S_{n-1} \\ B_n: & S_n \geq \sqrt{5} \end{cases}$

base case: $(n=2)$ $S_2 = \dfrac{S_1^2+5}{2\cdot S_1} = \dfrac{5^2+5}{2\cdot 5} = \dfrac{5^2+5}{10} = \dfrac{30}{10} = 3.$

indeed A_2: $3 \leq 5$ ✓

B_2: $3 \geq \sqrt{5}$ $\Leftrightarrow$ $9 \geq 5$ ✓

both holds.

induction step: Assume A_k, B_k holds for $2 \leq k \leq n$

now we show A_{n+1} and B_{n+1}

$A_{n+1} \Leftrightarrow S_{n+1} \leq S_n$

$\Leftrightarrow \dfrac{S_n^2+5}{2\cdot S_n} \leq S_n$

(given that $S_n > 0$, by B_n)

$\Leftrightarrow \quad S_n^2+5 \leq 2\cdot S_n^2$

$\Leftrightarrow \quad 5 \leq S_n^2$ guaranteed by B_n.

$\therefore A_{n+1}$ ✓.

$B_{n+1} \Leftrightarrow \quad S_{n+1} \geq \sqrt{5}$

$\Leftrightarrow \quad S_n^2+5 \geq \sqrt{5}\cdot 2S_n$.

$\Leftrightarrow \quad S_n^2 - 2\cdot\sqrt{5}\cdot S_n + 5 \geq 0$

$\Leftrightarrow \quad (S_n-\sqrt{5})^2 \geq 0$. automatically ✓.

Hence A_n, B_n holds for all n. (S_n) monotone decreasing, bounded below $\Rightarrow$ $\alpha = \lim S_n$ exists in $\mathbb{R}$.

(2). what is α? Since we have equality

$$S_{n+1} = \frac{S_n^2+5}{2\cdot S_n}$$

taking limit on both sides.

$$\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} \frac{S_n^2+5}{2\cdot S_n} = \frac{\lim(S_n^2+5)}{\lim(2\cdot S_n)} =$$

$\Rightarrow \quad \alpha = \dfrac{\alpha^2+5}{2\cdot\alpha} \Rightarrow 2\alpha^2 = \alpha^2+5$

$\Rightarrow \quad \alpha^2 = 5$

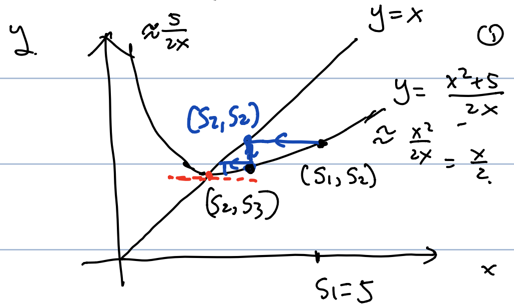
$\Rightarrow \quad \alpha$ can be only $+\sqrt{5}$, or $-\sqrt{5}$.

since all $S_n > 0$, α cannot be a negative number

$\Rightarrow \quad \alpha = \sqrt{5}$. #

General method to find the limits of a recursive seq.

$S_{n+1} = f(S_n)$. $\qquad f(x) = \dfrac{x^2+5}{2\cdot x}$



① Draw graph of $y = f(x)$, and $y = x$

② plot (S_1, S_2)

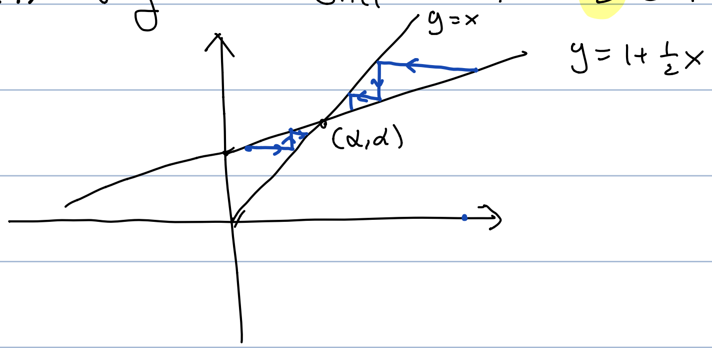
$\because S_2 = f(S_1)$

$\therefore (S_1, S_2)$ is on the graph $y = f(x)$.

③ the zigzag trajectory will lead you to the limiting point, which is the intersection of $y=x$ and $y=f(x)$

$\Rightarrow$ solve for $x = f(x)$

Ex (1) try $\quad S_{n+1} = 1 + \frac{1}{2} S_n$. $\qquad f(x) = 1 + \frac{1}{2}x$.



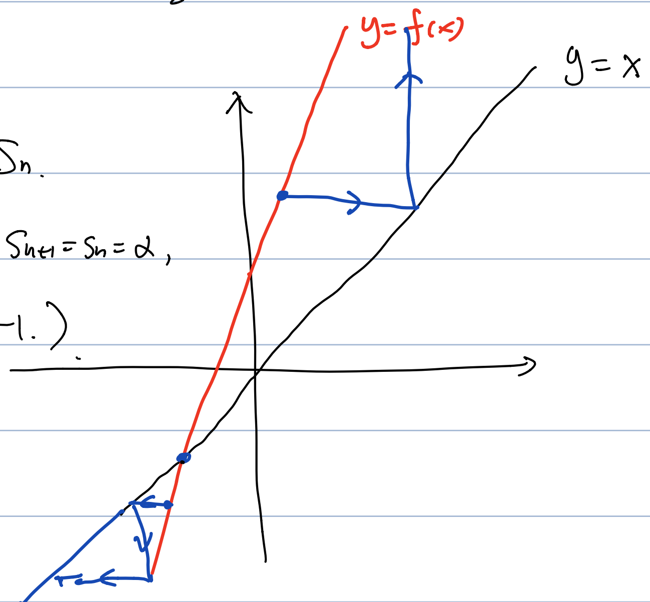
claim: for any S_1, $\quad \lim S_n = \alpha$. , where

$\alpha = 1 + \frac{1}{2}\alpha \Rightarrow \alpha = 2$.

(2) try $\quad S_{n+1} = 1 + 2S_n$.

(S_n) does not converge, (unless $S_{n+1} = S_n = \alpha$,

where $\alpha = 1 + 2\alpha \Rightarrow \alpha = -1$.)



§11 Sub sequence.

Def: Let (s_n) be a seq. let (n_k) be a strictly increasing seq in $\mathbb{N}$.

$n_1 < n_2 < n_3 < \cdots$

then we define a new seq.

$t_k := s_{n_k}$ for $k = 1, 2, \cdots$

(t_k) is called a subseq corresponding to (n_k),

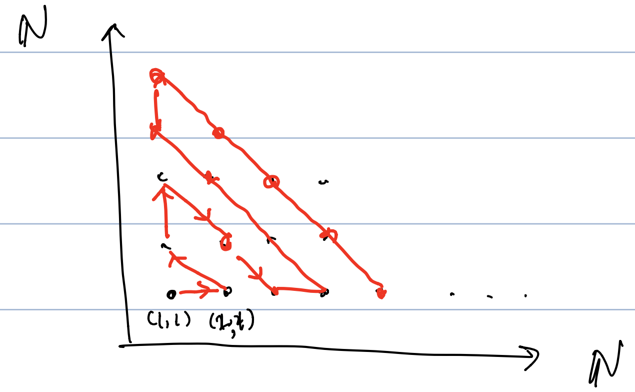
or $(s_{n_k})_k$

- sometimes, given an infinite subset $A \subset \mathbb{N}$, we can enumerate elements in A, as

$n_1 < n_2 < n_3 < \cdots$

then we get a subseq of (s_n), denoteds as $(s_n)_{n \in A}$.

Countable set:



hence $\mathbb{N} \times \mathbb{N}$ is countable.
$\parallel$
can be "counted".

- is $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ countable? Yes.

Pf: since $\mathbb{N}$ and $\mathbb{N}^2$ are countable, hence the product is countable.

- Lemma: (1) if sets A and B are countable, then

$$A \times B = \{(a,b) \mid a \in A,\ b \in B\}$$

is countable

(2) if A is countable, and $A' \subset A$, then A' is countable

$\square$

- Ex: $(S_n) = 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \cdots$

$\forall t \in \mathbb{R},\ 0 \le t \le 1$

it doesn't converge. But, we can take a subsequence that converge to t.

- $\frac{\sqrt{2}}{2} = \frac{1.414\cdots}{2} = 0.707\cdots$

one can get a $(t_n) = (0, 0.7, 0.70, 0.707, \cdots)$

then one can "embed" (t_n) into (S_n), by writing

$0.7 = \frac{7}{10}$, $\quad 0.70 = \frac{70}{100}$, $\quad \cdots\cdots$

then (t_n) is a subsequence of (S_n).

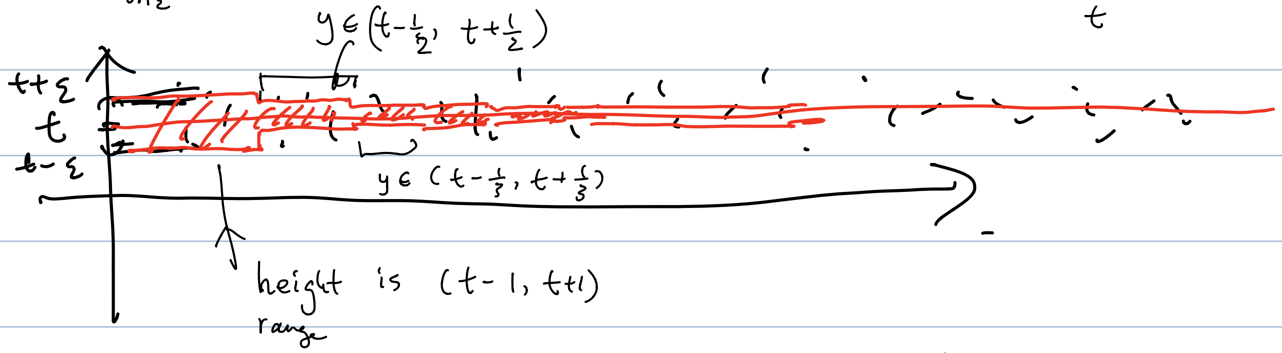
Thm: Let (S_n) be any sequence, and $t \in \mathbb{R}$.

then (S_n) has a subseq converge to t if and only if $\forall \varepsilon > 0$, the set $A_\varepsilon = \{ n \in \mathbb{N} \mid \underbrace{|S_n - t| < \varepsilon}_{S_n \in (t-\varepsilon,\ t+\varepsilon)} \}$ is infinite.

Pf: "only if" ✓.

"if": if we consider the subseq $(S_n)_{n \in A_\varepsilon}$, but

$(s_n)_{n \in A_\varepsilon}$ does not converge to t.



$A_\varepsilon = \{n \mid |s_n - t| < \varepsilon\}$

- $n_1 \in A_1$,

  $n_2 \in \{n > n_1\} \cap A_{\frac{1}{2}}$

  $n_3 \in \{n > n_2\} \cap A_{\frac{1}{3}}$.

  ⋮

$\Rightarrow \quad n_1 < n_2 < n_3 < \cdots$

$|s_{n_k} - t| < \frac{1}{k}$

hence $(s_{n_k})_k \to t$.

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(read Ross for the rigorous proof)