Today: ① common mistakes in HW1

#4. $\sqrt{2+\sqrt{2}}$ is not a rational number

we reduce to show that $\sqrt{2+\sqrt{2}}$ is not an integer. If it were an integer. then its a solution to $x^2 = 2+\sqrt{2} \Leftrightarrow (x^2-2)^2 = 2$.,

$$x^4 - 4x^2 + 2 = 0 \qquad x \text{ can only be } \pm 1, \pm 2.$$

Just plug in $x = \pm 1, \pm 2$ into the equation, and show ~~it~~ they are not sol'n. Hence $\sqrt{2+\sqrt{2}}$ ~~are~~ is not an integer.

② Recursive sequence, i.e. S_{n+1} is determined by S_n. (and possibly. n as well)

(P58 of Ross).

Ex: $S_1 = 5$, $\quad S_{n+1} = \dfrac{S_n^2 + 5}{2 \cdot S_n}$.

Q: (S_n) converges?

If so, converge to what?

Strategy: ① prove that S_n is a monotone descreasing sequence, and show that S_n is bounded below.

we will show that

$$A_n : \quad S_n \leq S_{n-1}. \qquad (n \geq 2)$$

$$B_n : \quad \sqrt{5} \leq S_n$$

- Suppose A_n, B_n holds, we are going to show A_{n+1}, B_{n+1} hold.

$$A_{n+1} \Leftrightarrow S_n \geq S_{n+1}$$

$$\Leftrightarrow S_n \geq \frac{S_n^2 + 5}{2 S_n}.$$

by B_n, we know $S_n > \sqrt{5} > 0$, hence we can multiply the above ineq by $2S_n$, a positive number, and get

$$\Leftrightarrow \quad 2S_n^2 \geq S_n^2 + 5$$

$$\Leftrightarrow \quad S_n^2 \geq 5$$

$$\Leftrightarrow \quad S_n \geq \sqrt{5}$$

this is provided by B_n, Hence. A_{n+1} holds.

Next, we show B_{n+1} holds.

$$B_{n+1} \Leftrightarrow S_{n+1} \geq \sqrt{5}$$

$$\Leftrightarrow \frac{S_n^2+5}{2S_n} \geq \sqrt{5}$$

$$\Leftrightarrow S_n^2+5 \geq \sqrt{5}\cdot 2S_n.$$

$$\Leftrightarrow S_n^2 - 2\cdot\sqrt{5}\cdot S_n + 5 \geq 0$$

$$\Leftrightarrow (S_n - \sqrt{5})^2 \geq 0$$

this holds automatically.

- To check. statement holds for $n=2$, we compute

$$S_2 = \frac{S_1^2+5}{2\cdot S_1} = \frac{5^2+5}{2\cdot 5} = \frac{30}{10} = 3.$$

indeed

$$\begin{cases} A_2: & S_1 \geq S_2 \\ B_2: & 3 \geq \sqrt{5} \qquad (\because 9 \geq 5) \end{cases}$$

they holds for $n=2$.

② what is the limit? let $\alpha = \lim_{n\to\infty} S_n$, then.

α satisfies

$$\alpha = \frac{\alpha^2+5}{2\cdot\alpha} \qquad (*)$$

Indeed, by taking limit on both sides of

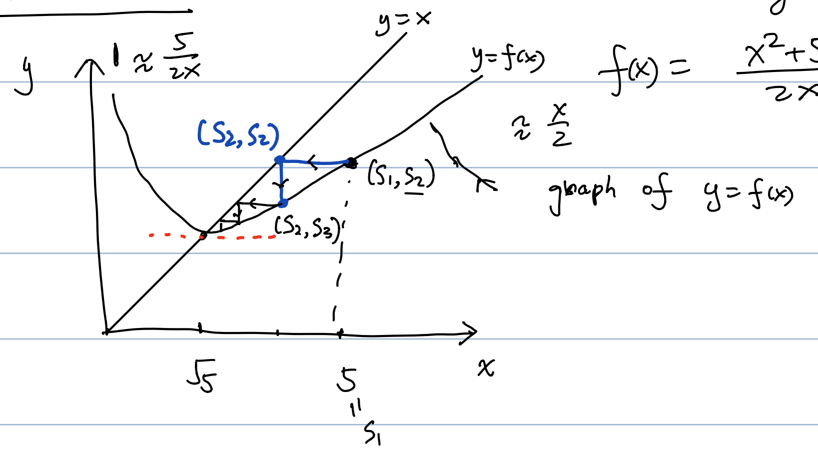
$$S_{n+1} = \frac{S_n^2+5}{2S_n}.$$

we get $(*)$. $(*) \Rightarrow \alpha^2 = 5 \Rightarrow$ possible value $\alpha = \sqrt{5}$ or $\alpha = -\sqrt{5}$.

since $S_n \geq \sqrt{5}$, $\therefore$ $\alpha = \sqrt{5}$.

General Method (to visualize and analyse recursive seq.)

$f(x) = \frac{x^2+5}{2x}$, $S_{n+1} = f(S_n)$



- to create this trajectory, we do "horizontal to the diagonal, go vertical to the graph"

---

§11 sub sequence.

Definition: Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of real number. Given a strictly increasing seq of indices

$$n_1 < n_2 < n_3 < \cdots < n_m < \cdots$$

We define the corresponding sub sequence as

$$t_k := S_{n_k}$$

$(t_k)_k$ is called a sub seq of $(S_n)_n$.

Sometimes, we write $(S_{n_k})_k$ for the subseq.

Ex: $S_n = (-1)^n \cdot \left(\frac{1}{n}\right)$. $(S_n) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \cdots\right.$

if we take

$(n_1, n_2, \cdots) = (2, 4, 6, \cdots,)$ $\qquad n_k = 2k$.

then.

$$t_k = S_{n_k} = S_{2k} = (-1)^{2k} \cdot \left(\frac{1}{2k}\right) = \frac{1}{2k}.$$

$$(t_1, t_2, \cdots) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots\right)$$

with limit in $\mathbb{R}$

Lemma 1: if (S_n) is convergent, then any subsequence converges to the same point.

Pf: Let $\alpha = \lim_n S_n$. Let (S_{n_k}) be a subsequence. $\forall \varepsilon > 0$, we need to find a $K > 0$, s.t. if $k > K$, then $|S_{n_k} - \alpha| < \varepsilon$. Since $S_n \to \alpha$, we have an N, s.t. $\forall n > N$, $|S_n - \alpha| < \varepsilon$. So, we may take K to be a large enough integer, s.t. $n_K > N$. , then $\forall k > K$, $n_k > n_K > N$, thus $|S_{n_k} - \alpha| < \varepsilon$. #.

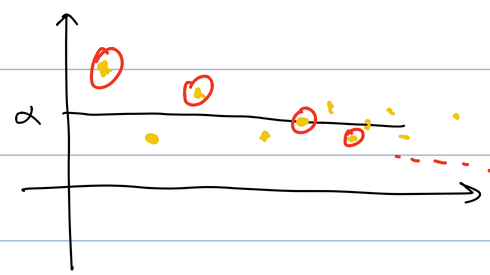
Lemma 2: If $\alpha = \lim_n S_n$ exist in $\mathbb{R}$., then there exists a subsequence that is monotone.

Pf: Consider 3 subset of indices.

$A = \{n \in \mathbb{N} \mid S_n > \alpha\}$

$B = \{n \in \mathbb{N} \mid S_n = \alpha\}$

$C = \{n \in \mathbb{N} \mid S_n < \alpha\}$.



since $\mathbb{N} = A \sqcup B \sqcup C$ ($\sqcup$ = disjoint union).

at least one of A, B, C are infinite.

case (a) : if B is infinite, then the subseq corresponds to B is enough.

case (b) : if A is infinite. We want to create a decreasing sequence. We do this inductively:

Let $A_0 = A$. pick $n_1 \in A_0$.

Define $A_1 = \{ n \in A_0 \mid n > n_1, \quad s_n \leq s_{n_1} \}$.
then pick $n_2 \in A_1$.

Define $A_2 = \{ n \in A_1 \mid n > n_2, \quad s_n \leq s_{n_2} \}$.
then pick $n_3 \in A_2$

claim : A_i are all infinite. (exercise).

so the process can go on forever. and we get

$n_1 < n_2 < n_3 < \cdots$

and $s_{n_1} \geq s_{n_2} \geq s_{n_3}$,

(all $s_{n_k} > \alpha$).

$\{ n : s_n < \alpha \}$.

case (c) : if C is infinite. we can build an increasing seq.

#.

Lemma 3 : Let (s_n) be any sequence. Then for any $t \in \mathbb{R}$ (s_n) $\underbrace{\text{has a subseq converges to } t}_{\text{"subconverges to } t\text{"}}$ $\Leftrightarrow$ $\forall \varepsilon > 0$, the set $\{ n \mid |s_n - t| < \varepsilon \}$ is infinite.

Pf: $\Rightarrow$ Let (S_{n_k}) converges to t. Then $\forall \varepsilon > 0$,

$\exists K > 0$, s.t. $|S_{n_k} - t| < \varepsilon$, $\forall k > K$.

then $\{n_k \mid k > K\}$ is an infinite set,

contained in $\{n \mid |s_n - t| < \varepsilon\}$.

$\Leftarrow$ We are going to build n_k iteratively.

Let $\varepsilon_k = \frac{1}{k}$. for $k = 1, 2, \cdots$.

Then, let $n_1 \in \{n \mid |S_n - t| < \varepsilon_1\}$.

Assume $n_1, \cdots, n_{k-1}$ are constructed, s.t.

$n_1 < n_2 < \cdots < n_{k-1}$, $|S_{n_i} - t| < \varepsilon_i$.

we can construct n_k as follows.

pick any $n_k \in \{n \mid \underline{n > n_{k-1}}, \underline{|S_n - t| < \varepsilon_k}\}$.

By induction, we get a seq $n_1 < n_2 < \cdots$.

since $|S_{n_k} - t| < \varepsilon_k = \frac{1}{k}$. We have $S_{n_k} \to t$ as $k \to \infty$. #

Ex: $(S_n) = ($0, 1,

0, 0.1, 0.2, 0.3, ⋯ , 0.9, 1,

0, 0.01, 0.02, ⋯ ⋯ , 0.99, 1,

0, 0.001, 0.002, ⋯ , 0.999, 1, ⋯

⋮ $)$

say $\alpha \in (0,1)$ is $\alpha = 0.a_1 a_2 \cdots$ $a_i \in \{0, \cdots, 9\}$

then define (t_n) ~~S_n~~ $= 0.a_1 a_2 \cdots a_n$

n digits.

t_n is a subseq of S_n.