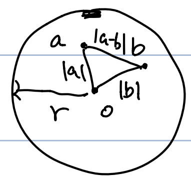
Last time: §10 Monotone Seq and Cauchy Seq.

- Thm: Bounded monotone seq has limit.
- Thm: A sequence (a_n) converges if and only if (a_n) is Cauchy.

<u>Def</u>: (Cauchy Seq.) (a_n) is Cauchy, if $\forall \varepsilon > 0, \exists N > 0$, s.t. $\forall n, m > N$, we have $|a_n - a_m| < \varepsilon$.

<u>Last time</u>: (a_n) convergent $\Rightarrow$ (a_n) Cauchy.

basic idea:



$|a-b| < |a-o| + |b-o| = r + r = 2r.$

if $\forall n > N$, $|a_n - \alpha| < \varepsilon$

then $\forall n, m > N$, $|a_n - a_m| < |a_n - \alpha| + |\alpha - a_m| < \varepsilon + \varepsilon = 2\varepsilon.$

To show $\Leftarrow$, we need some preparation:

lim sup, lim inf.

<u>Lemma</u>: Let (a_n) be a bounded seq, then.

$\lim a_n$ exists $\Leftrightarrow$ $\liminf a_n = \limsup a_n$.

<u>Pf</u> (we will do today)

<u>Today</u>: Let (a_n) be a bounded seq.

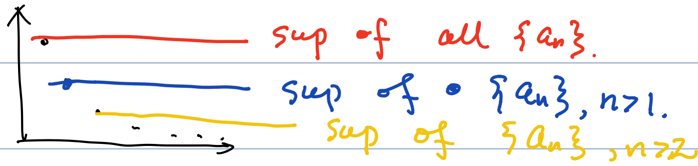
<u>Lemma</u>: $\forall \varepsilon > 0, \exists N > 0$, s.t. $\forall n > N$, we have

$$a_n < \limsup a_n + \varepsilon.$$

<u>rmk</u>: lim sup is not a sup of any set!

<u>Ex</u>: $a_n = \frac{1}{n}$,

$$\limsup a_n := \lim_{N \to \infty} \left(\sup_{n \geq N} (a_n) \right) = \lim_{N \to \infty} (a_N) = \lim \frac{1}{N} = 0.$$



here. $\limsup (\frac{1}{n}) = 0$, it is smaller than any a_n.

"an ε of room":

Ex:

- to prove $a=b$, suffice to prove, for any $\varepsilon > 0$, $|a-b| < \varepsilon$.
- if we know before hand $a \le b$, then to prove $a=b$, just need $b-a < \varepsilon$, $\forall \varepsilon > 0$.

  $\Leftrightarrow$ $b < a+\varepsilon$ $\forall \varepsilon > 0$.

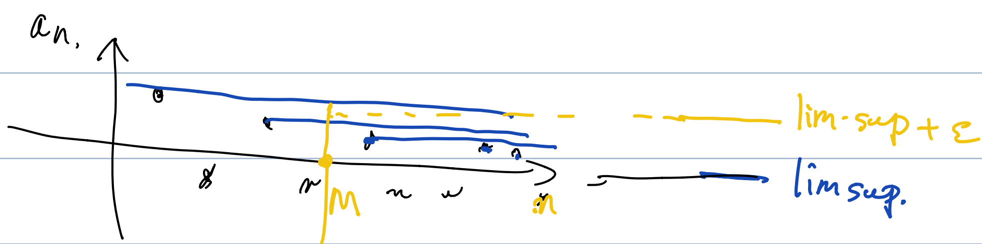
- $a_n \to \alpha$, means, $\forall \varepsilon > 0$, we consider interval $(\alpha-\varepsilon, \alpha+\varepsilon)$, and we have the tail part of the seq (a_n) inside this interval.

Pf of Lemma: Let $S_N = \sup\limits_{n \ge N} a_n$. Since $\lim S_N = \limsup a_n$ exists, say equal α, $\forall \varepsilon > 0$, $\exists M > 0$, s.t. $\forall N > M$, $|S_N - \alpha| < \varepsilon$.

$$\alpha - \varepsilon < S_N \le \alpha + \varepsilon \qquad \forall N > M.$$

$\Rightarrow$ $\forall n \ge N, \forall N \ge M$. $a_n \le S_N < \alpha + \varepsilon$.

$\Rightarrow$ $\forall n \ge M$, $a_n < \alpha + \varepsilon$. #.



Similarly, if (a_n) bounded, then $\forall \varepsilon > 0$, $\exists N > 0$. s.t.

$$\forall n > N, \quad a_n > \liminf a_n - \varepsilon.$$

Now, back to.

Lemma: Let (a_n) be a bounded seq, then.

$$\lim a_n \text{ exists} \iff \liminf a_n = \limsup a_n.$$

Pf: $\Leftarrow$ Let $\alpha = \limsup a_n$, then $\alpha = \liminf a_n$ by assumption.

By previous Lemma, $\forall \varepsilon > 0$, $\exists N_1 > 0$. s.t.

$$\forall n > N_1, \quad a_n < \limsup a_n + \varepsilon.$$

$\exists N_2 > 0$, s.t.

$$\forall n > N_2, \quad a_n > \liminf a_n - \varepsilon.$$

Then, take $N = \max(N_1, N_2)$,

$$\forall n > N, \quad \alpha - \varepsilon < a_n < \alpha + \varepsilon \qquad (*).$$

This is precisely the requirement that $a_n \to \alpha$.

#.

Thm: Let (a_n) be any sequence, then

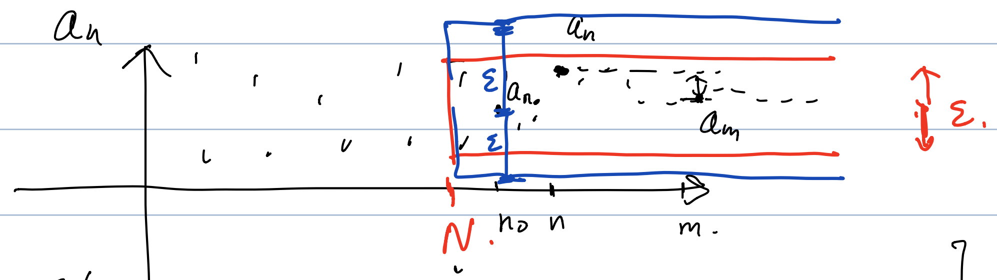
$$(a_n) \text{ is Cauchy} \iff (a_n) \text{ converges.}$$

Pf: $\Leftarrow$ by triangle inequality.

$\Rightarrow$ We will prove by showing $\limsup$ and $\liminf$

exists and are equal.

First remark, (a_n) is Cauchy $\Rightarrow$ (a_n) is bounded.



$\forall \varepsilon > 0$, $\exists N$, s.t.

$|a_n - a_m| < \varepsilon \qquad \forall\, n, m > N.$ } by Cauchy condition.

if we fix $n_0 > N$, then $\underline{a_{n_0} - \varepsilon} < a_m < \underline{a_{n_0} + \varepsilon}$ $\forall\, m > N$.

hence the tail part of (a_n) $(n \geq N)$ is bounded.

hence the sequence itself (a_n) is bounded.

(compare with the proof that (a_n) converges $\Rightarrow$ (a_n) is bounded)

- Since $\forall\, m > N$, we have

$$a_{n_0} - \varepsilon < a_m < a_{n_0} + \varepsilon,$$

$$\Rightarrow \quad \limsup a_n < a_{n_0} + \varepsilon$$

$$a_{n_0} - \varepsilon < \liminf a_n$$

$$\Rightarrow \quad \limsup a_n - \varepsilon < a_{n_0} < \liminf a_{n_0} + \varepsilon.$$

$$\Rightarrow \quad \limsup a_n < \liminf a_n + 2\varepsilon. \qquad (*)$$

$(*)$ is true for any $\varepsilon > 0$. $\Rightarrow$

$$\limsup a_n \leq \liminf a_n.$$

On the otherhand, for any bounded seq $\liminf a_n \leq \limsup a_n$.

$\Rightarrow$ $\liminf a_n = \limsup a_n$ $\Rightarrow$ $\lim a_n$ exists and equals $\liminf a_n$.

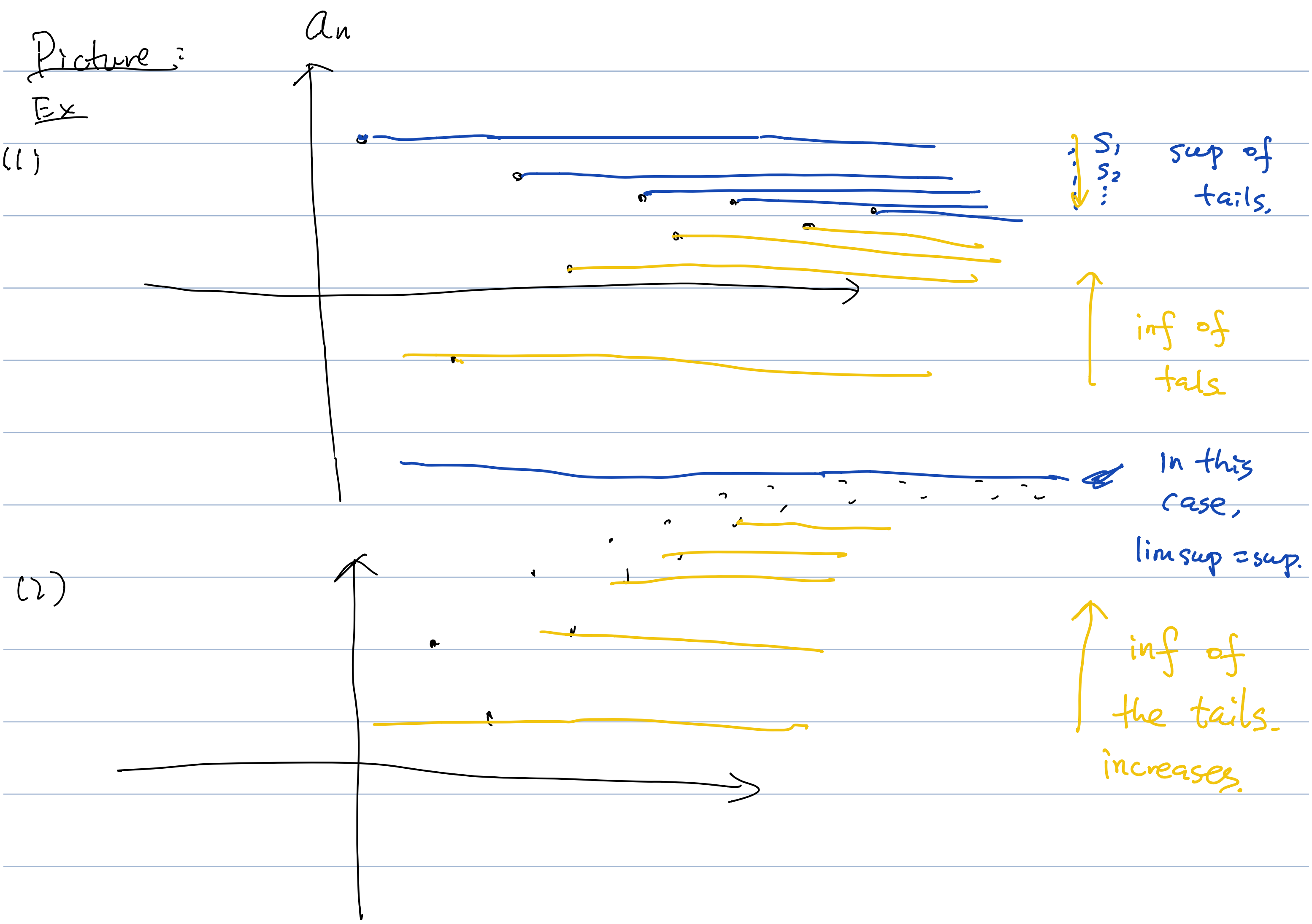
HW. 9.9 (c): if (s_n), (t_n) seq, $s_n \leq t_n$, then. $\lim s_n \leq \lim t_n$.

Given this result, if we define $S_N = \sup_{n \geq N} a_n$, $I_N = \inf_{n \geq N} a_N$, then $I_N \leq S_N$, hence

$\lim\inf a_n = \lim I_N \leq \lim S_N = \lim\sup a_n$

Picture:

Ex



☆ Recursive Sequene:

$\underbrace{S_1 = 5}$, $\quad S_n = \dfrac{S_{n-1}^2 + 5}{2 S_{n-1}}$ $\quad \forall n \geq 2.$

- if S_n has a limit. , i.e. if $\lim S_n = \alpha$.

then, I claim, if $\alpha \neq 0$. then.

$$\alpha = \frac{\alpha^2 + 5}{2\alpha}.$$

Pf: if $\lim S_n = \alpha \neq 0$, then.

$$\lim_{n\to\infty} \frac{S_n^2 + 5}{2S_n} = \frac{\alpha^2 + 5}{2\alpha}$$

and $\lim S_n = \alpha$ hence $\alpha = \dfrac{\alpha^2 + 5}{2\alpha}$. #

$\Rightarrow 2\alpha^2 = \alpha^2 + 5 \Rightarrow$ possible value of α. $\alpha = \pm\sqrt{5}$.

Q: • does $\lim S_n$ exist ?

- if so, is $\alpha = 0$?
- if $\alpha \neq 0$, then $\alpha = +\sqrt{5}$ or $-\sqrt{5}$?

Claim: $\underbrace{\sqrt{5} <}_{A_n} S_{n+1} \underbrace{<}_{B_n} S_n \underbrace{\leq 5}_{C_n}$. $\quad \forall n \geq 1$.

$S_1 = 5$, $\quad S_2 = \dfrac{5^2 + 5}{2 \cdot 5} = \dfrac{30}{10} = 3$,

- A_n, B_n, C_n are true, for $n = 1$.
- Assume $A_{n-1}, B_{n-1}, C_{n-1}$, are true

we want to show A_n, B_n, C_n.

To show B_n holds:

$$S_{n+1} < S_n \iff \frac{S_n^2+5}{2\cdot S_n} < S_n.$$

$$\Leftarrow \begin{cases} S_n > 0. & \left(\text{true by } A_{n-1}.\right) \\ S_n^2+5 < 2\cdot S_n^2. \end{cases}$$

$$\Leftarrow \quad S_n > \sqrt{5} \quad \left(\text{statement of } A_{n-1}\right)$$

- To show C_n holds, i.e. $S_n \leq 5$, we use monotonicity and previous bound on S_{n-1}:

$$\underbrace{S_n \leq S_{n-1}}_{B_n} \underbrace{\leq 5}_{C_{n-1}} \Rightarrow \underbrace{S_n \leq 5}_{C_n}$$

- To show A_n hold, i.e. $\sqrt{5} \leq S_{n+1}$, we need

$$\sqrt{5} \leq \frac{S_n^2+5}{2S_n}.$$

By B_{n-1}, $S_n > \sqrt{5} > 0$, hence suffice to prove that

$$S_n^2+5 \geq 2\sqrt{5}\, S_n.$$

$$\iff (S_n-\sqrt{5})^2 \geq 0$$

which always holds. $\therefore A_n$ ✓.

(to clear denominator in an inequality, we need to be careful with signs.)