Last time: Ross §10. Monotone Sequences and Cauchy Sequence

Thm: any bounded monotone sequence is convergent.

Thm: (not finished last time)

$$(a_n)_n \text{ is Cauchy} \iff (a_n)_n \text{ is convergent.}$$

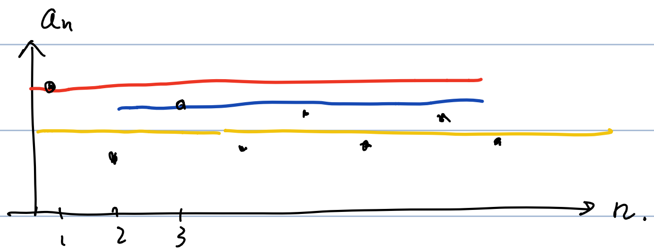
last time $\Leftarrow$ ✓. Today $\Rightarrow$.

Def: lim inf and lim sup. Let $(a_n)_n$ be a seq in $\mathbb{R}$.

$$\limsup a_n := \lim_{N\to\infty}\left(\sup_{n\geq N}\{a_n\}\right).$$

$$\liminf a_n := \lim_{N\to\infty}\left(\inf_{n\geq N} a_n\right).$$

idea:

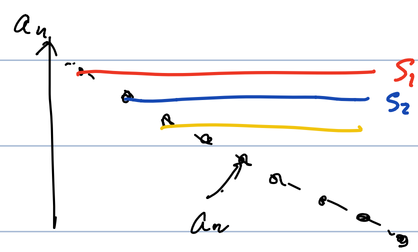


$\sup\{a_n\}$ = in red line.

$\sup_{n>1}\{a_n\}$ = in blue line

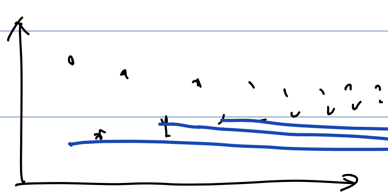
- if define $S_N = \sup_{n\geq N}\{a_n\}$, then (S_N) is a decreasing seq.

S_N may not be bounded from below:



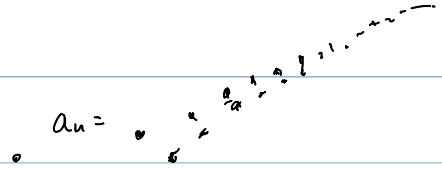
- if (a_n) is bounded, then (S_N) is bounded.

- Similarly, we discuss the liminf.

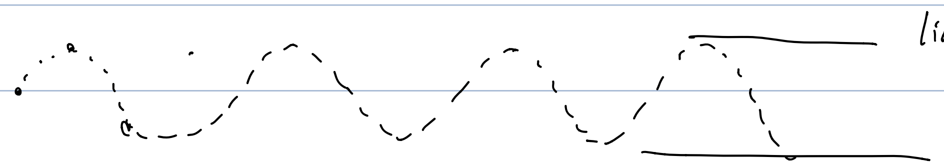


. Rmk: if we allow the notion of $\lim = +\infty$ or $\lim = -\infty$, then $\limsup$ exists in $\mathbb{R}$ or $= -\infty$

$\liminf$ exists in $\mathbb{R}$ or $= +\infty$.



- a monotone increasing seq has two fates
  - (1) if it's bounded, then its limit exist.
  - (2) if it's unbounded, then $\lim a_n = +\infty$



- "Give oneself an ε of room": - Terry Tao

  Ex: (1) if we want to prove $a = b$, one way to prove it is $|b-a| < \varepsilon$ for any $\varepsilon > 0$.

- **Lemma**: if (a_n) is a bounded sequence, then its $\limsup$, $\liminf$ exists.

Pf: Say $-M < a_n < M$ for all n. Then, $-M < S_N < M$ for all N, where $S_N = \sup\limits_{n \geq N} \{a_n\}$. Since $S_N \searrow$ monotone, and bounded, hence $\lim S_N$ exists, i.e. $\limsup a_n$ exist.

#.

- **Lemma**: if (a_n) is a bounded sequence, and $\alpha_+ = \limsup a_n$, then for any $\varepsilon > 0$, $\exists N$, such that $\forall n > N$, we have

$$a_n < \alpha_+ + \varepsilon.$$

- similarly, $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n > N$, $a_n > \liminf a_n - \varepsilon$.

Pf: Let $S_N = \sup_{n > N} a_n$. Since $S_n \to \alpha_+$ as $n \to \infty$,

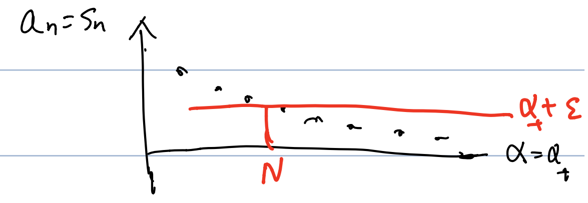
then $\forall \varepsilon > 0, \exists N > 0$, s.t. $\forall n > N$, $\alpha_+ - \varepsilon < S_n < \alpha_+ + \varepsilon$.

Since $S_n \geq a_m$, for all $m \geq n$, we have

$$a_m < \alpha_+ + \varepsilon \qquad \forall m \geq n, \quad n > N$$

$$\Leftrightarrow \quad a_m < \alpha_+ + \varepsilon \qquad \forall m > N. \qquad \#$$

Ex:



Thm: Let (a_n) be a bounded seq. Then

$$\lim a_n \text{ exists} \iff \limsup a_n = \liminf a_n$$

$\alpha = \lim a_n$

Pf: $\Rightarrow$ we show $\limsup a_n = \lim a_n$. Suffice to show that

$\forall \varepsilon > 0$, $\left| \limsup a_n - \lim a_n \right| < \varepsilon$.

[actually, if $S_n = \sup_{m \geq n} a_m$, then $S_n \geq a_n$. then

$\lim S_n \geq \lim a_n$. by Ex 9.9 (c). Hence $\limsup a_n \geq \lim a_n$.

$\forall \varepsilon > 0$, $\exists N > 0$. s.t. $\forall n > N$, $|a_n - \alpha| < \varepsilon$,

$\Rightarrow \quad \underline{S_N} < \alpha + \varepsilon \Rightarrow \overset{\forall M \geq N}{S_M} < \alpha + \varepsilon. \Rightarrow \lim S_N < \alpha + \varepsilon$.

i.e. $\lim a_n \leq \lim S_N < \lim a_n + \varepsilon$

Since this holds for any $\varepsilon > 0$, ~~this~~ $\lim S_N = \lim a_n$.

$\boxed{\text{Let } \alpha = \limsup a_n = \liminf a_n.}$

$\Leftarrow$ $\forall \varepsilon > 0$, $\exists N_1 > 0$, s.t. $\forall n > N_1$,

$$a_n < \limsup a_n + \varepsilon = \alpha + \varepsilon.$$

$\exists N_2 > 0$, s.t. $\forall n > N_2$.

$\alpha - \varepsilon = \liminf a_n - \varepsilon < a_n$

Let $N = \max(N_1, N_2)$. Then $\forall n > N$,

$$\alpha - \varepsilon < a_n < \alpha + \varepsilon \qquad \#.$$

Now, proof that (a_n) converges $\Leftarrow$ (a_n) is a Cauchy seq.

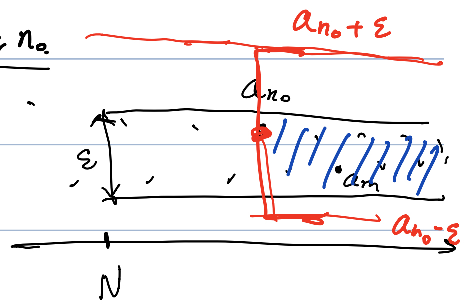
[recall: (a_n) is a Cauchy seq, if $\forall \varepsilon > 0$, $\exists N > 0$. s.t. $\forall n, m > N$. $|a_n - a_m| < \varepsilon$.]

Pf: We are going to show that $\limsup a_n = \liminf a_n$.

Suffice to show, $\forall \varepsilon > 0$, $\limsup a_n - \liminf a_n < 2\varepsilon$.

By Cauchy property of (a_n), $\exists N > 0$, s.t. $\forall n, m \geq N$, $|a_n - a_m| < \varepsilon$. Fix $n = n_0 > N$, then. $\forall m \geq n_0$

$$a_{n_0} - \varepsilon < \underline{\underline{a_m}} < a_{n_0} + \varepsilon \qquad \underline{\forall m \geq n_0}$$



$\Rightarrow$ $\limsup a_n < a_{n_0} + \varepsilon$

$a_{n_0} - \varepsilon < \liminf a_n$.

$\Rightarrow$ $\limsup a_n < \liminf a_n + 2\varepsilon$. $\underline{(*)}$.

since also. $\limsup a_n \geq \liminf a_n$, and $(*)$ is true for all $\varepsilon > 0$, $\Rightarrow$ $\limsup a_n = \liminf a_n$.

$\Rightarrow$ $\lim a_n$ exists.

Ex: (1) Let $\alpha = 0.999\cdots9\cdots$; Let $a_n = 0.\overbrace{99\cdots9}^{n \text{ of them}}$,

then a_n is a monotone sequence, and it is bounded, i.e. $a_n < 1 \quad \forall n.$ $\overset{\text{M.C.T.}}{\Longrightarrow}$ a_n has a limit. And

$\lim a_n = 1$ $\quad \because \quad 1 - a_n = \frac{1}{10^n}$, (hence $\forall \varepsilon > 0$, if we

let $N = \log \varepsilon / \log 10$, then $\forall n > N$, $|1 - a_n| < \varepsilon$.).

hence. $0.99\cdots = 1$.

(2). Let (S_n) be a seq. s.t.

$$|S_{n+1} - S_n| < 2^{-n} \quad \text{for all } n \in \mathbb{N}.$$

$$\Rightarrow \quad |S_n - S_{n+k}| \leq |S_n - S_{n+1}| + |S_{n+1} - S_{n+2}| + \cdots + |S_{n+k-1} - S_{n+k}|$$

$$\leq 2^{-n} + 2^{-n-1} + \cdots + 2^{-(n+k-1)}$$

$$= 2^{-n}\left(1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots + 2^{-(k-1)}\right)$$

$$\leq 2^{-n} \cdot 2.$$

$\Rightarrow$ (S_n) is a Cauchy sequence.

indeed, $\forall \varepsilon > 0$, $\exists\, 2^{-N} < \varepsilon$. so $\underset{m}{\forall n > N}$. $|a_n - a_m| < 2^{-N}$.