Today : series §14 §15.

- Series. : $\sum_{n=1}^{\infty} a_n$. (if we only care about convergence (not the actual numbers), we can write $\sum_n a_n$)

partial sum . $S_n = \sum_{j=1}^{n} a_n$.

we say $\sum_n a_n$ converges iff the seq (S_n) converge.

- Cauchy Condition for series :
  - The series $\sum_n a_n$ satisfies Cauchy condition, if $\forall \varepsilon > 0$, $\exists$ integer $N > 0$. s.t. $\forall$ integer $n, m > N$,

$$\left| \sum_{j=n}^{m} a_n \right| < \varepsilon.$$

- Cauchy of $\sum_n a_n$ $\Leftrightarrow$ Cauchy of (S_n) $\Leftrightarrow$ convergence of $\sum_n a_n$, (S_n).

- Cor : If $\sum_n a_n$ converges, then $\lim a_n = 0$. (necessary condition).

Pf : by Cauchy condition, take $n = m$. Then $\forall \varepsilon > 0$, $\exists N$, s.t. $\forall n > N$, we have $|a_n| < \varepsilon$.

- Ex (geometric series) $\sum_{n=0}^{\infty} a \cdot r^n$

$= a(1 + r + r^2 + \cdots)$

claim. $= a \cdot \frac{1}{1-r}$ if $|r| < 1$

Pf : Form the partial sum.

$$S_n = a(1 + \cdots + r^n) = a \frac{1 - r^{n+1}}{1 - r}$$

since $|r| < 1$, $\lim r^{n+1} = 0$. $\lim S_n = a \cdot \frac{1}{1-r}$

recall : $\forall x \neq 1$

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Pf : $(1-x)(1 + x + \cdots + x^n)$

$= 1 + x + \cdots + x^n - x - x^2 - \cdots - x^{n+1}$

$= 1 - x^{n+1}$ #.

- **Comparison Test**:
  - Suppose $\sum_n a_n$ converges $\cancel{< \infty}$, $a_n > 0$.

    Suppose we have a series $\sum_n b_n$, $|b_n| < a_n$.

    Then $\sum_n b_n$ converges.

Pf: We verify that $\sum_n b_n$ satisfies the Cauchy condition. Since $\sum_n a_n$ converges, hence $\forall \varepsilon > 0$, $\exists N > 0$. s.t. $\forall n, m \geqslant N$, we have

$$\sum_{j=n}^{m} a_j = \left| \sum_{j=n}^{m} a_j \right| < \varepsilon .$$

Thus, $\forall n, m \geqslant N$

$$\left| \sum_{j=n}^{m} b_j \right| \leqslant \sum_{j=n}^{m} |b_j| \leqslant \sum_{j=n}^{m} a_j < \varepsilon .$$

#.

Def. (absolute convergence): We say a series $\sum_n b_n$ converges absolutely, if $\sum_n |b_n|$ converges.

Ex: (converges, but not absolutely converge).

- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \cdots$$

$$= \frac{1}{2} + \frac{1}{3 \times 4} + \frac{1}{5 \times 6} \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n)}$$

(later, we will see $\sum \frac{1}{n^2}$ converges. using integral test.)

Q: $\sum (-1)^{n+1} \frac{1}{\sqrt{n}}$ converges?

Prop: if $a_n > 0$, $\sum_n a_n = +\infty$. And if $b_n > a_n$, then $\sum_n b_n = +\infty$

pf: Let (s_n), (t_n) be partial sums of $\sum_n a_n$ and $\sum b_n$, then. $t_n \geq s_n$ $\forall n \in \mathbb{N}$.

Since $\lim s_n = +\infty$. hence $\lim t_n = +\infty$. #.

Root Test, Ratio test:

Recall in §12, given any sequence (a_n) $(a_n \neq 0)$, then.

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \underset{\alpha.}{\overset{\shortparallel}{\limsup |a_n|^{\frac{1}{n}}}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|.$$

Prop: Given a series $\sum_n a_n$. Assume $a_n \neq 0$.

Let $\alpha = \limsup |a_n|^{\frac{1}{n}}$. Then.

(1) if $\alpha > 1$., then $\sum_n a_n$ diverges.

(2) if $\alpha < 1$, then $\sum_n a_n$ converges absolutely.

(3) if $\alpha = 1$, $\sum_n a_n$ could converge or diverge.

Pf: (2). Since $\alpha < 1$, then there exists an $\varepsilon > 0$, s.t. $\alpha + \varepsilon < 1$. Pick such an ε. Since for any $\delta > 0$, $\exists N > 0$, s.t. $\forall n > N$. $|a_n|^{\frac{1}{n}} < \alpha + \delta$.

Hence, for this chosen ε, we have an $N > 0$. s.t.

$$|a_n|^{\frac{1}{n}} < \alpha + \varepsilon \quad \forall n > N.$$

$$\Longleftrightarrow \quad |a_n| < (\alpha+\varepsilon)^n \quad \forall n > N.$$

since $0 < \alpha + \varepsilon < 1$, hence $\sum_n (\alpha+\varepsilon)^n$ converges.

By the comparison test. $\sum_{n=N+1}^{\infty} |a_n|$ converges.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

(1). $\alpha = \limsup \underline{|a_n|^{\frac{1}{n}}} > 1$. $\exists$ a subseq $(a_{n_k})_k$ of the seq $(a_n)_n$, such that

$$|a_{n_k}|^{\frac{1}{n_k}} \longrightarrow \alpha.$$

Hence, if we pick an $\varepsilon > 0$, s.t. $\alpha - \varepsilon > 1$, then $\exists N > 0$, s.t. $\forall k > N$.

$$|a_{n_k}|^{\frac{1}{n_k}} > \alpha - \varepsilon > 1.$$

$$\Rightarrow \quad |a_{n_k}| > 1 \qquad \forall k > N.$$

This implies (a_n) does not converge to 0.
Hence $\sum_n a_n$ is not convergent.

<u>Ratio test</u>: consider $\sum_n a_n$. $(a_n \neq 0)$.

(1) if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_n a_n$ converges absolutely.

<u>Pf</u>: $\because \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$.

$\therefore \limsup |a_n|^{\frac{1}{n}} < 1 \quad \therefore \sum_n a_n$ conv. abs. #

(2). if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_n a_n$ diverges.

<u>Pf</u>: $\because \limsup |a_n|^{\frac{1}{n}} \geq \liminf \left|\frac{a_{n+1}}{a_n}\right|$.

$\therefore \limsup |a_n|^{\frac{1}{n}} > 1$.

$\therefore \sum_n a_n$ diverges.

Ross §14.1

<u>Ex</u>: (a) $\sum \frac{n^4}{2^n}$ (b) $\sum \frac{2^n}{n!}$

(c) $\sum n^2/3^n$. (f) $\sum_{n=2}^{\infty} \frac{1}{\log n}$

(a) Using root test. $\left(\frac{n^4}{2^n}\right)^{\frac{1}{n}} = (n^4)^{\frac{1}{n}} \cdot \frac{1}{2}$

$\lim n^{\frac{1}{n}} = 1$. $\therefore \lim (n^{\frac{1}{n}})\cdot(n^{\frac{1}{n}}) = 1$, ----

$\forall k \in \mathbb{N}$. $\lim (n^{\frac{1}{n}})^k = 1$, $\therefore (n^4)^{\frac{1}{n}} = (n^{\frac{1}{n}})^4 \to 1$.

$\therefore |a_n|^{\frac{1}{n}} \to 1 \cdot \frac{1}{2} < 1$.

$\therefore \alpha = \lim (a_n)^{\frac{1}{n}} = \frac{1}{2}$ $\sum a_n$ converges.

(b). $\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \frac{2}{1} \cdot \frac{1}{n+1}$

$= \frac{2}{n+1} \longrightarrow 0$

since $\limsup \left|\frac{a_{n+1}}{a_n}\right| = 0$, by ratio test,

$\sum a_n$ converges.

$\sum_n \frac{2^n}{n!} = e^2$.

(C) $a_n = \frac{n^2}{3^n}$. Using root test, we have.

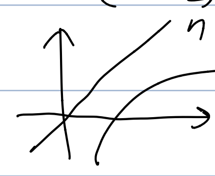
$(a_n)^{\frac{1}{n}} = \frac{1}{3} \cdot (n^2)^{\frac{1}{n}} \longrightarrow \frac{1}{3} < 1$

$\therefore$ converges. absolutely.

Harmonic Series

(d) $\sum_{n=2}^{\infty} \frac{1}{\log n} > \sum_n \frac{1}{n} = +\infty$ $\because \log n < n$ $(n \geq 2)$

by comparison test. $\therefore \frac{1}{\log n} > \frac{1}{n}$



$\sum \frac{1}{\log n}$ is divergent.

$x < e^x$ for $x > 0$

$e^x = 1 + x + \frac{x^2}{2!} \cdots$

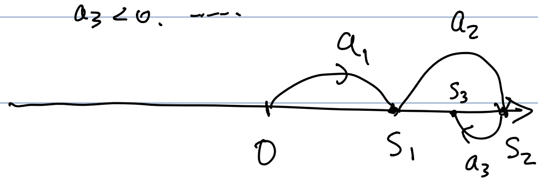
Alternating Series

$$\sum_{n=1} (-1)^{n+1} \cdot a_n \qquad a_n > 0.$$

Thm: If $a_1 \geq a_2 \geq a_3 \geq \cdots$, $a_n \geq 0$, and $\lim a_n = 0$.

Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge.

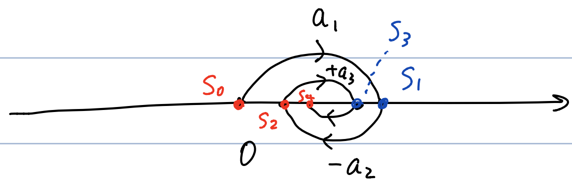
Ex: $a_1 > 0$, $a_2 > 0$, $a_3 < 0$. ......



S_n : location at time n.

a_n : the step at time n.

$a_1 - a_2 + a_3 - a_4 \cdots$



Pf: Define $S_n = \sum_{j=1}^{n} a_n$.

then claim $(S_{2n})_n$ forms an increasing bounded seq. ✓

$(S_{2n-1})_n$ forms a decreasing bounded seq. ✓

Indeed,

- $S_{2n+2} - S_{2n} = (-1)^{2n+2+1} a_{2n+2} + (-1)^{2n+1+1} a_{2n+1}$
  $= -a_{2n+2} + a_{2n+1} \geq 0.$
- $S_{2n+1} - S_{2n-1} = (-1)^{2n+1+1} a_{2n+1} + (-1)^{2n+1} a_{2n}$
  $= +a_{2n+1} - a_{2n} \leq 0.$

- to show $S_{2n} \leq S_1$, we write

$$S_{2n} = S_1 + \underbrace{(-a_2 + a_3)}_{\leq 0} + \underbrace{(-a_4 + a_5)}_{\leq 0} + \cdots + \underbrace{(-a_{2n})}_{\leq 0}.$$

$\therefore$ $S_{2n} \leq S_1$

- similarly, $S_{2n+1} \geq 0$,

$$S_{2n+1} = \overbrace{(a_1 - a_2)}^{\geq 0} + (a_3 - a_4) + \cdots + \overbrace{(a_{2n-1} - a_{2n})}^{\geq 0} + \underbrace{a_{2n+1}}_{\geq 0}.$$

$\geq 0.$

· Hence $\lim S_{2n} = \alpha$, and $\lim S_{2n+1} = \beta$ exists.

To show $\alpha = \beta$, we take

$$\beta - \alpha = \lim_n S_{2n+1} - \lim_n S_{2n}$$

$$= \lim_n (S_{2n+1} - S_{2n})$$

$$= \lim_n a_{2n+1} = 0.$$

#

Ex: · $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$, since $\lim \left| (-1)^n \frac{1}{\sqrt{n}} \right| = 0$

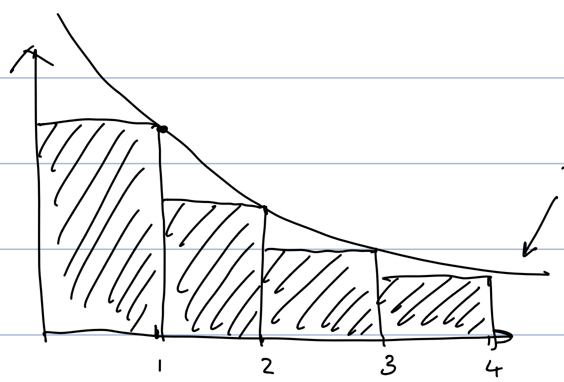
and $\left(\frac{1}{\sqrt{n}}\right)$ is decreasing,

$\therefore$ the series converge.

· $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\log n}$ converges.

Integral test:

Thm: $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ if $p > 1$.



$a_n = \frac{1}{n^p}$.

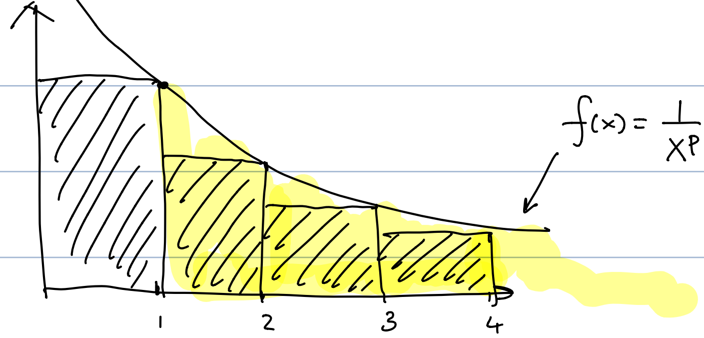
shaded area $= \sum_n \frac{1}{n^p}$.

$$\int_{x=1}^{\infty} \frac{1}{x^p}\, dx$$

$=$ yellow area.

$\geq \quad a_2 + a_3 + \cdots$

$$= \sum_{n=2}^{\infty} \frac{1}{n^p}.$$



$\therefore \quad \sum_n \frac{1}{n^p}$ converges iff $\int_1^{\infty} \frac{1}{x^p}\, dx$

$$\int_1^{\infty} \frac{1}{x^p}\, dx = \int_1^{\infty} d\left(\frac{x^{-p+1}}{-p+1}\right) = \left.\frac{x^{-p+1}}{-p+1}\right|_1^{\infty} = 0 - \frac{1}{-p+1}$$

$$= \frac{1}{p-1} < \infty.$$

$$x^{-p} = \left(\frac{x^{-p+1}}{-p+1}\right)'$$

$\therefore$ if $p > 1$. then $\sum \frac{1}{n^p} < \infty$. #.