Today: §14, §16 Series.

- Series is an infinite sum.

$$\sum_{n=1}^{\infty} a_n.$$

- partial sum $S_n = \sum_{i=1}^{n} a_i$

$S_1 = a_1$, $S_2 = a_1 + a_2$, ...

$m > n$, $S_m - S_n = \sum_{i=n+1}^{m} a_i$

- Def.: a series converge to α if the corresponding partial sum converges.

- Cauchy condition for series convergence: (sufficient & necessary condition).

$\forall \varepsilon$, $\exists N > 0$, s.t. $\forall n, m > N$,

$$\left| \sum_{i=n+1}^{m} a_i \right| < \varepsilon.$$

- Absolute convergence of series $\sum_n a_n$:

definition: if $\sum |a_n| < \infty$, we say $\sum a_n$ converges absolutely.

- Ex: (1) geometric series. $\sum_{n=0}^{\infty} a \cdot r^n$

converges if $|r| < 1$.

$$S_n = \sum_{i=0}^{n} a \cdot r^i$$

$$= a(1 + r + \cdots + r^n) \qquad (r \neq 1)$$

$$= a \cdot \frac{1 - r^{n+1}}{1 - r}$$

$\lim S_n < \infty$ if $|r| < 1$. In this case.

$\lim_n r^{n+1} = 0$. $\lim S_n = a \cdot \frac{1}{1-r}$.

$|r| > 1$, Ex: $1+2+4+8 \cdots$ $\sum_{n=0}^{\infty} 2^n$ diverges.

Various test for series convergence:

① Comparison test

(a)• Suppose $\sum_n a_n < \infty$, $a_n > 0$. And suppose $b_n \in \mathbb{R}$. $|b_n| \leq a_n$, then $\sum_n b_n < \infty$

(b)• Suppose $\sum_n a_n = \infty$, $a_n > 0$. And suppose $b_n \geq a_n$, then $\sum_n b_n = \infty$.

Pf: (a). $\left|\sum_{i=n}^{m} b_i\right| \leq \sum_{i=n}^{m} |b_i| \leq \sum_{i=n}^{m} a_i$.

$\forall \varepsilon > 0$, consider $\sum_n a_n$, we have $N > 0$. s.t.

$\forall n, m > N$, $\sum_{i=n}^{m} a_i < \varepsilon \Rightarrow \left|\sum_{i=n}^{m} b_i\right| < \varepsilon$.

$\Rightarrow \sum_n b_n$ converges.

(b). let (S_n) be the partial sum of $\sum_n a_n$.

(t_n) ———————— of $\sum_n b_n$.

then $t_n \geq S_n$ $\forall n$. Since S_n diverges to $+\infty$.

hence $t_n \nearrow +\infty$.

② Ratio test: $\sum_n a_n$.

(testing for absolute convergence).

- if $\limsup |a_{n+1}/a_n| < 1$, then $\sum_n |a_n|$ converge.
- if $\liminf |a_{n+1}/a_n| > 1$. then $\sum_n a_n$ diverges.
- otherwise. the test gives no information.

③ Root test.

Let $\sum_n a_n$ be a series, let $\alpha = \limsup |a_n|^{\frac{1}{n}}$.

Then $\sum_n a_n$ (i) converges absolutely, if $\alpha < 1$

(ii). diverges, if $\alpha > 1$.

(iii). if $\alpha = 1$. no info.

Pf: (i). Since $\alpha < 1$, then $\exists\, \varepsilon > 0$, s.t. $\alpha + \varepsilon < 1$.

And since $\alpha = \limsup \underline{|a_n|}^{\frac{1}{n}}$, $\therefore\ \exists N > 0$, s.t.

$|a_n|^{\frac{1}{n}} < \alpha + \varepsilon$, $\forall n > N$.

$\iff \quad |a_n| < (\alpha+\varepsilon)^n \quad \forall n > N$.

$$\sum_n |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

$$\le \sum_{n=1}^{N} |a_n| + \underbrace{\sum_{n=N+1}^{\infty} (\alpha+\varepsilon)^n}_{= (\alpha+\varepsilon)^{N+1} \cdot \frac{1}{1-(\alpha+\varepsilon)}}$$

$$< \infty$$

if S_n is a seq, with $\alpha = \limsup S_n < \infty$.

then. $\alpha = \lim a_n$. $\qquad a_n = \sup\{S_m : m \geq n\}$.

sup of the tail of S_n.

then $\forall \varepsilon > 0$, $\exists N > 0$. s.t.

$$|a_n - \alpha| < \varepsilon \qquad \forall n > N.$$

$$\Leftrightarrow \quad -\varepsilon < a_n - \alpha < \varepsilon \qquad \forall n > N.$$

$$\Leftrightarrow \quad \alpha - \varepsilon < \underline{a_n < \alpha + \varepsilon} \qquad \forall n > N$$

$$\Rightarrow \quad \sup\{S_m : m \geq n\} < \alpha + \varepsilon \qquad \forall n > N.$$

$$\Rightarrow \quad S_m < \alpha + \varepsilon. \qquad \forall m > N+1.$$

(ii). If $\limsup |a_n|^{\frac{1}{n}} = \alpha > 1$, then. there is a subseq of $(|a_n|^{\frac{1}{n}})_n$, that converges to α, Denote this subseq. as. $(|a_n|^{\frac{1}{n}})_{n \in A}$, for some. index subset $A \subseteq \mathbb{N}$. Since $|a_n|$ does not converge to. 0, $\sum_n a_n$ is divergents.

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(equivalently, $a_n \to 0$)

<u>Lemma</u>: if $\sum_n a_n$ converges, then. $|a_n| \to 0$.

<u>Pf</u>: $\because$ $|a_n| = |S_n - S_{n-1}|$. , $\quad S_n = \sum_{j=1}^{n} a_n$.

$\therefore$ by Cauchy condition of convergence.,

$\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n, m > N$.

$$|S_n - S_m| < \varepsilon.$$

take. $n = m+1$, and get $|a_n| < \varepsilon$. $\forall n > N+1$.

thus $|a_n| \to 0$.

Warning: $|a_n| \to 0$ doesn't imply $\sum_n a_n$ converges.

i.e. $\sum_n \frac{1}{n} = +\infty$.

Given the root test, we can now prove the claim on ratio test. Recall, given any seq a_n ($a_n \neq 0$), we have,

(*) $\underline{\liminf \left|\frac{a_{n+1}}{a_n}\right|} \le \liminf |a_n|^{\frac{1}{n}} \le \underline{\limsup |a_n|^{\frac{1}{n}}} \le \underline{\limsup \left|\frac{a_{n+1}}{a_n}\right|}$

- if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\limsup |a_n|^{\frac{1}{n}} < 1$. hence $\sum_n |a_n|$ converges.

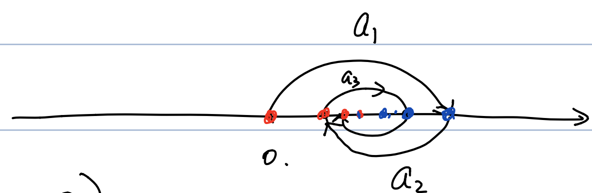
- if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\limsup |a_n|^{\frac{1}{n}} > 1$. hence $\sum_n a_n$ diverges.

<u>Alternating Series test</u>:

<u>Thm</u>: let $a_1 \geq a_2 \geq a_3 \geq \cdots$ be a montono decreasing series, $a_n \geq 0$. And assuming $\underline{\lim a_n = 0}$. Then.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 \cdots$$

converges.



<u>Pf</u>: let $S_n = \sum_{j=1}^{n} a_j$. ($S_0 = 0$).

$S_0, S_2, S_4 \cdots$ forms an increasing seq,

$S_1, S_3, S_5, \cdots$ forms a decreasing seq.
odd.

Indeed. $S_{2n+2} - S_{2n} = a_{2n+2} \cdot (-1)^{2n+2+1} + a_{2n+1} (-1)^{2n+1+1}$

$= a_{2n+1} - a_{2n+2} \geq 0.$

$$S_{2n+1} - S_{2n-1} = a_{2n+1} - a_{2n} \le 0.$$

One can check that $(S_{2n})_n$ and $(S_{2n+1})_n$ are bounded. Hence they have limits.

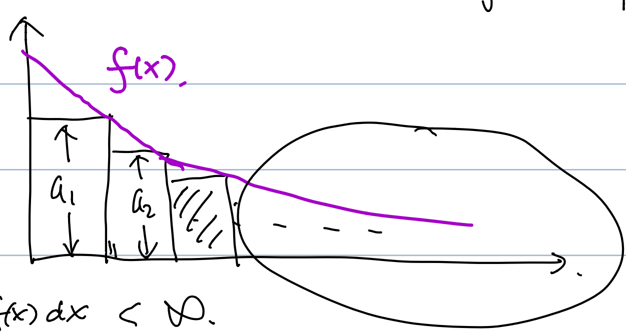
Let $\alpha = \lim_n S_{2n}$, $\beta = \lim (S_{2n+1})_n$.

$$\beta - \alpha = \lim_n S_{2n+1} - \lim_n S_{2n} = \lim_n (S_{2n+1} - S_{2n})$$

$$= \lim_n a_{2n+1} = 0.$$

$\therefore \lim_n S_n$ exists.

<u>Integral test</u>:
- replace Σ_n by $\int_x (\cdots) dx$,
- works for positive sequences.



$a_n = f(n)$.

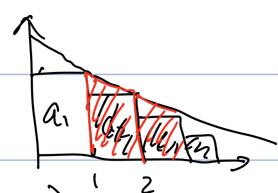
if $\int_1^\infty f(x)\,dx < \infty$.

$\Sigma_n a_n$ = Area under the columns $\le$ Area under the curve

$= \int_1^{+\infty} f(x)\,dx < \infty$

<u>Thm</u>: $\Sigma_n \frac{1}{n^p} < \infty$ if $p > 1$.

Pf: $\Sigma_{n=2}^{\infty} \frac{1}{n^p} \le \int_1^\infty \frac{1}{x^p} dx$



$= \int_1^\infty d\left(\frac{1}{[-(p-1)]x^{p-1}}\right)$

$$= \frac{1}{-(p-1)\, x^{p-1}} \Big|_1^{\infty} = \frac{1}{p-1} < \infty$$

Hence $\sum_n \frac{1}{n^p} < \infty$.

Ex: $\sum_{n=2}^{\infty} \frac{1}{n \cdot (\log n)^2} < \infty$ $\qquad (\log 1 = 0)$

consider integral $\int^{+\infty} \frac{1}{x (\log x)^2}\, dx$.

$$= \int^{\infty} \frac{1}{(\log x)^2}\, d(\log x).$$

change variable. let $u = \log x$. then as x runs from a to $+\infty$, where $a > 10$, then $\log x$ runs from $\log a$ to $+\infty$.

$$= \int^{\infty} \frac{1}{u^2}\, du. \quad < \infty$$

$\equiv$

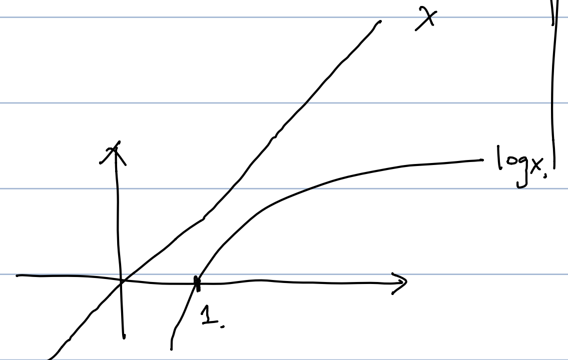
Ex. $\sum \frac{2^n}{n!}$, consider $\frac{a_n}{a_{n-1}} = \frac{2^n/n!}{2^{n-1}/(n-1)!} = \frac{2}{n} \to 0$.

by ratio test, it is convergent.

- $\sum \frac{2^n}{\sqrt{n!}}$, $\frac{a_n}{a_{n-1}} = \frac{2}{\sqrt{n}} \to 0$.

- $\sum \frac{1}{\log n}$.

$\sum \frac{1}{n} = +\infty$



for any $n \in \mathbb{N}$, $\log n < n$.

$\Rightarrow \frac{1}{\log n} > \frac{1}{n}$ $\quad n \geq 2$.

by comparison test. $\sum \frac{1}{\log n} = +\infty$.

ratio test $\rightarrow$ $\left|\frac{a_{n+1}}{a_n}\right| = \sqrt{\frac{n}{n+1}} \rightarrow 1$.

$\cdot \quad \sum (-1)^{n+1} \frac{1}{\sqrt{n}}$. $\leftarrow$ root test. $|a_n|^{\frac{1}{n}} = \left(\frac{1}{\sqrt{n}}\right)^{\frac{1}{n}}$

Alternating series test. $\quad = \left(\underbrace{n^{\frac{1}{n}}}\right)^{-\frac{1}{2}}$

$1 > \frac{1}{\sqrt{2}} > \frac{1}{\sqrt{3}} \cdots$ $\quad \longrightarrow 1$.

$\therefore$ convergent.

(by continuity of function $f(x) = \frac{1}{\sqrt{x}}$ at $x=1$.)

$\cdot$ For any $\theta \in (0, 2\pi)$ (i.e. $\theta \neq 0$.)

$$\sum \frac{\cos(n\theta)}{n} \quad \text{and} \quad \sum \frac{\sin(n\theta)}{n}.$$

are. 皆 convergent.

(Hint: $e^{i\theta} = \cos\theta + i \cdot \sin\theta$.

assume convergent.

consider $\sum \frac{e^{in\theta}}{n} = f(\theta)$.

$f'(\theta) = \sum_{n=1}^{\infty} i \cdot e^{in\theta} = i \cdot \frac{1}{1 - e^{i\theta}}$

$\hookrightarrow$ find primitive of $f'(\theta)$. -----)