Metric Space and Topology.

- (S, d) metric space.

$$\begin{cases} S : \text{ set} \\ d : \quad S \times S \to \mathbb{R}. \\ \quad \cdot\ d(x,y) = d(y,x) \\ \quad \cdot\ d(x,y) \geq 0, \quad = 0 \text{ iff } x = y \\ \quad \cdot\ d(x,y) + d(y,z) \geq d(x,z). \end{cases}$$

- Open subsets in S.

__Def__: $\mathcal{U} \subset S$ is __open__ iff

$\forall\, p \in \mathcal{U}, \quad \exists\, \delta > 0, \quad \text{s.t.} \quad B_\delta(p) \subset \mathcal{U}.$

$B_\delta(p)$ = open ball centered at p, radius δ

$= \{ q \in S \mid d(p,q) < \delta \}$

$B^\times_\delta(p)$ = punctured open ball. $= B_\delta(p) \setminus \{p\}$.

$S \setminus E$.

$\shortparallel$

__Def__: $E \subset S$ is __closed__ iff. E^c is open

i.e,

$E \subset S$ is closed iff $\forall\, x \notin E, \quad \exists\, \delta > 0, \quad \text{s.t.}$

$B_\delta(x) \cap E = \emptyset$.

__Properties__: Open	Closed.
• $S, \emptyset$ are open.	• $S, \emptyset$ are closed.
• arbitrary union of open subsets is open.	• arbitrary intersection of closed subsets is closed.
• finite intersection of open is open	• finite union of closed set is closed.

__Ex__: in $\mathbb{R}$. Example of __open sets__:

(a,b), $(a, +\infty)$, $(-\infty, a)$, $(-\infty, +\infty) = \mathbb{R}$

Example of closed set:

$[a,b]$, $[a,+\infty)$, $(-\infty, a]$, $\{a\}$, $\mathbb{R}$.

Cantor set: $I_0 = [0,1]$,

$I_1 = [0,\frac{1}{3}] \cup [\frac{2}{3}, 1]$

$I_2 = [0,\frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

$\vdots$

$I_0 \supset I_1 \supset I_2 \supset \cdots$.

$I := \bigcap I_n$ intersection of closed sets is still closed.

• Def: (closure) Let $E \subset S$ be any subsets. We define the closure $\bar{E} = \bigcap \{F \mid F \subset S \text{ closed set}, F \supset E\}$.

(interior) $E^\circ = \bigcup \{U \mid U \subset S \text{ open}, U \subset E\}$.

$= \{p \in E \mid \exists \delta > 0, B_\delta(p) \subset E\}$.

(boundary) $\partial E = \bar{E} \setminus E^\circ$.

Ex: $E = [a,b)$, $\bar{E} = [a,b]$, $E^\circ = (a,b)$.

$\partial E = \{a,b\}$. (same is true, if $E = (a,b)$, $[a,b]$, $(a,b]$)

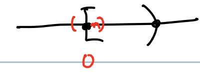
• Def (limit point) Let $E \subset S$. A point $p \in S$ is a limit point of E, if $\forall \varepsilon > 0$, $\exists q \in E$, $q \neq p$, such that $d(p,q) < \varepsilon$.

E' = the set of lim points of E.

Ex: $E = [0,1)$. then $\{1\}$ is a limit point.

E

$E' = [0, 1]$



Ex: $E = \{ \frac{1}{n} \mid n \in \mathbb{N} \} = \{ 1, \frac{1}{2}, \frac{1}{3}, \cdots \}$

$E' = \{0\}$.

Prop: $\overline{E} = E \cup E'$. (proof as exercise).

(part of Ross Prop 13.9)

Prop: $E \subset S$ is closed

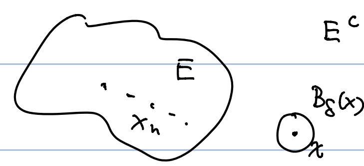
$\Leftrightarrow$ $\forall$ convergent sequence (x_n), in S, $x = \lim x_n$.

if (x_n) are in E, then the limit x is in E.

Ex: $E = (0, 1]$ is not closed, $\because$ the sequence $(\frac{1}{n})$ has limit 0, $\frac{1}{n} \in E$, but $0 \notin E$.

Pf: $\Rightarrow$ Suppose E is closed. and suppose there exist $x_n \to x$, with $x_n \in E$, $x \notin E$. Then, $x \in E^c$, E^c is open, hence $\exists\, \delta > 0$, s.t. $B_\delta(x) \subset E^c$, i.e. $B_\delta(x) \cap E = \emptyset$.

This contradict with $x_n \to x$, and $x_n \in E$.



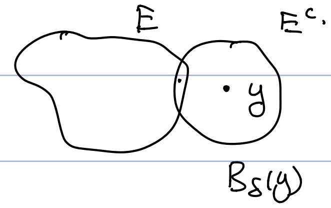
$\Leftarrow$ Suppose $\forall\, x_n \to x$, $x_n \in E$, we have $x \in E$.

Suppose E is not closed, i.e. E^c is not open.

then $\exists\, y \in E^c$, s.t. $\forall\, \delta > 0$, $B_\delta(y) \not\subset E^c$, i.e.

$B_\delta(y) \cap E \neq \emptyset$. Then, we can construct a seq (y_n), by choosing $y_n \in B_{\frac{1}{n}}(y) \cap E$. Then.

$0 < d(y_n, y) < \frac{1}{n}$. Hence, $y_n \to y$.



By hypothesis, y has to be in E.

This contradict with $y \notin E$.

• Compact subset.

<u>Def</u> (open cover): Let $E \subset S$;
An open cover of E is a collection of open sets $\{G_\alpha\}_{\alpha \in A}$
s.t. $E \subset \bigcup_\alpha G_\alpha$.

compact subsets in $\mathbb{R}$.

- $[a,b]$, $a, b \in \mathbb{R}$
- $\{a\}$ compact.

non-compact.

- $\mathbb{R}$
- $(0,1)$

<u>Def</u> (compact) $K \subset S$ is compact, if for any open cover of K, we can find a finite subcover.

<u>Ex</u>: • $K \subset S$, <u>finite subset</u>.

<u>Pf</u>: say $K = \{x_1, \cdots, x_N\}$. Suppose we are given an open cover $\{G_\alpha\}_{\alpha \in A}$, then $x \in K \subset \bigcup_\alpha G_\alpha$.
then $\exists\, G_{\alpha(x)}$, s.t. $G_{\alpha(x)} \ni x$. Thus, we can use

$$G_{\alpha(x_1)} \cup G_{\alpha(x_2)} \cup \cdots \cup G_{\alpha(x_N)}$$

is a finite subcover. More precisely, let

$$A' = \{\alpha(x_1), \cdots, \alpha(x_N)\}.$$

then $\{G_\alpha\}_{\alpha \in A'}$ is ~~a~~ a finite subcover.

<u>Ex</u>: • $K = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\} \cup \{0\} \subset \mathbb{R}$ is compact.

Pf: Let $\{G_\alpha\}$ be an open cover of K.

Then, $\exists\, G_{\alpha_0}$ that, $0 \in G_{\alpha_0}$. $\exists\, \delta > 0$, s.t. $B_\delta(0) \subset G_{\alpha_0}$. Thus. $G_{\alpha_0} \ni \frac{1}{n}$, for all $\frac{1}{n} < \delta$.

Thus. there are only finitely many points in K, not covered by G_{α_0}. Say $\frac{1}{N} < \delta$, then. $\forall\, n \leq N$, we let G_{α_n} cover the point $\frac{1}{n}$, then.

$$\{G_{\alpha_1}, G_{\alpha_2}, \cdots, G_{\alpha_N}, G_{\alpha_0}\}$$

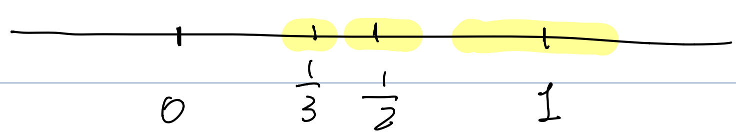
is a finite subcover of K.

Ex: $K = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is not compact.

Pf: for each $n \in \mathbb{N}$, let $\delta_n = \frac{1}{2} \cdot \min\left(\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right)$

then $B_{\delta_n}(\frac{1}{n}) \cap K = \{\frac{1}{n}\}$.

Then $K \subset \bigcup_{n \in \mathbb{N}} B_{\delta_n}(\frac{1}{n})$, there is no proper subcover. of $\{B_{\delta_n}(\frac{1}{n})\}_n$



Ex: $\mathbb{R}$ is non compact.

$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n-1, n+1)$ no finite subcover.

Thm 1 $K \subset \mathbb{R}^n$.

(Heine-Borel) K compact $\Leftrightarrow$ K is closed and bounded.

Pf: ① compact $\Rightarrow$ bounded. Let $K \subset (S, d)$ be a compact subset. pick any point $p \in S$, and consider $\{ B_n(p) : n \in \mathbb{N} \}$, this covers S. hence also covers K.

$$K \subset S = \bigcup_{n \in \mathbb{N}} B_n(p).$$

By compactness of K, there is a finite subcover.

$$K \subset B_{n_1}(p) \cup B_{n_2}(p) \cup \cdots \cup B_{n_M}(p).$$

say $n_1 < n_2 < \cdots < n_M$, then

$$K \subset B_{n_M}(p).$$

Hence K is bounded. , $\forall x, y \in K$.

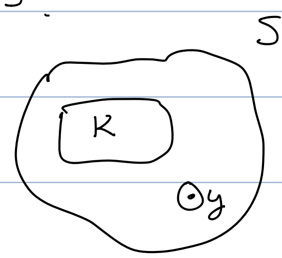
$$d(x,y) \leq d(x,p) + d(y,p) \leq 2 \cdot n_M.$$

② compact $\Rightarrow$ closedness. $K \subset (S, d)$ compacts.

To show K is closed, we need to show $\forall y \in K^c$, $\exists \delta > 0$, such that $B_\delta(y) \cap K = \emptyset$.

$\forall n \in \mathbb{N}$, $\mathcal{U}_n = \{ q \in S \mid d(q, y) > \frac{1}{n} \} = \left(\overline{B_{\frac{1}{n}}(y)} \right)^c$

$$\bigcup_n \mathcal{U}_n = S \setminus \{y\} \supset K$$



since K is covered by a finitely many such $\mathcal{U}_n$,

$$K \subset U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_M} \qquad n_1 < n_2 < \cdots$$

$$\Rightarrow \quad K \subset U_{n_M}.$$

$$\Rightarrow \quad K \cap \underline{B_{\frac{1}{n_M}}(y)} = \phi.$$

③. **Lemma**: closed subset in a compact set is compact. i.e. if $E \subset S$ closed, $K \subset S$, compact, and $E \subset K$, then E is also compact.

Pf: let $\{G_\alpha\}$ be an open cover of E. then $\{G_\alpha\} \cup \{E^c\}$ is an open cover of K. Hence,

$$K \subset E^c \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_N}.$$

for some collection $\alpha_1, \cdots, \alpha_N$. (taking $E \cap (\cdots)$ on both sides)

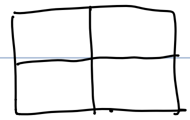
$$\Rightarrow \quad E \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_N}.$$

④. n-cells in $\mathbb{R}^n$ are compact,

(an n-cells is $\underbrace{[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]}_{= I}$. for some a_i, b_i.)

Pf: · Suppose the n-cell I is not compact, then. $\exists$ an open cover $\{G_\alpha\}$, that doesn't admit any finite subcover.

- divide I in each direction by $\frac{1}{2}$,

$I =$ 

get 2^n parts. (smaller ~~k~~-cells)

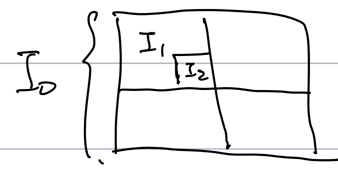
then. $\{G_\alpha\}$ is an open cover for any of them.

<u>claim</u>: there exist at least one of the smaller n-cell, for which $\{G_\alpha\}$ doesn't admit a finite subcover.

Let I_1 be one such "bad" n-cells.

- $I = I_0 \supset I_1 \supset I_2 \supset \cdots$.

$\operatorname{diam}(I_n) = \frac{1}{2^n} \cdot \operatorname{diam}(I_0)$.



- <u>claim</u>: $\bigcap I_n \neq \phi$.

<u>Pf</u>: (exercise)

- $x \in \bigcap I_n$. $\exists G_\alpha$ open, that covers x.

$\Rightarrow G_\alpha$ cover I_n for n large enough.

we have a contradiction that, the open cover $\{G_\alpha\}$ of I_n doesn't admit a finite subcover. #.