Topology of metric space.

- (S, d) be a metric space. S set. $d : S \times S \to \mathbb{R}$.
  - $d(x,y) \geq 0$, $d(x,y) = 0$ iff $x = y$
  - $d(x,y) = d(y,x)$
  - $d(x,y) + d(y,z) \geq d(x,z)$ $\quad \forall x, y, z \in S$

- Let $E \subset S$ any subset.
  - $p \in E$ is an interior point of E if $\exists\, \delta > 0$, s.t.

    $B_\delta(p) = \{ q \in S \mid d(p,q) < \delta \} \subset E$.
  - E° = the set of interior points of E. "interior of E"

Def : $E \subset S$. is an open subset of S.

if $E = E^\circ$. i.e. $\forall\, p \in E$, $\exists\, \delta > 0$. s.t. $B_\delta(p) \subset E$.

Prop : Let (S, d) be metric space

- S, ϕ. are open
- if $\{G_\alpha\}$ is a collection of open sets. then

  $\bigcup_\alpha G_\alpha$ is open.
- if $\{G_i\}_{i=1}^N$ open, then $\bigcap_{i=1}^N G_i$ open.

Def : $E \subset S$ is a closed subset of S, if

the complement $E^c = S \setminus E$ is open.

emptyset.

Prop : (1) S, ϕ are closed.

(2). if $\{F_\alpha\}$ is a collection of closed sets.

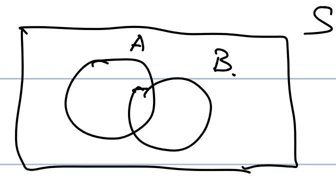
$\bigcap_\alpha F_\alpha$ is closed.

(3). if $\{F_i\}_{i=1}^N$ is ________ closed sets.

then $\bigcup_{i=1}^N$ is closed.

De Morgan Law : if $A, B \subset S$ subsets. then.

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c.$$



Ex : S any set. $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$

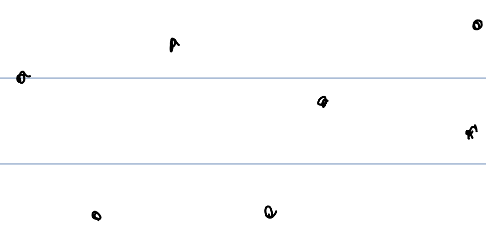
- $\forall p \in S$, $\{p\}$ is open. since

$\{p\} = B_{\frac{1}{2}}(p).$ = open ball of radius $\frac{1}{2}$ centered at p.

$= \{ q \in S \mid d(q,p) < \frac{1}{2} \}.$

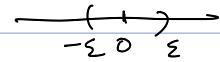
- $\{p\}$ is also closed., since

$\{p\}^c = \bigcup_{q \in S \setminus \{p\}} B_{\frac{1}{2}}(q)$ is a union of open balls, hence is open.

$S =$ 

- Def: (limit points). Let $E \subset S$, $p \in S$ is a limit point of E, iff $\forall \delta > 0$, $\overset{\times}{B_\delta}(p) = \{ q \in S \mid q \neq p,\ d(q,p) < \delta \}.$ intersects E ~~non~~ non empty., i.e. $\exists q \in E$, $q \neq p$, $d(q,p) < \delta$.

Let $E' =$ set of limit points of E.



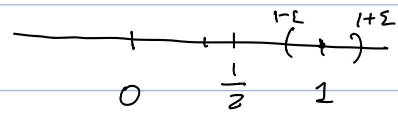
Ex: $S = \mathbb{R}$, $E = \{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots\}$. $(-\varepsilon, 0) \cup (0, \varepsilon)$.

then 0 is a limit point of E. $\because$ $\forall \varepsilon > 0$. $B^{\times}_{\varepsilon}(0) \cap E \neq \phi$.

$\frac{1}{2}$ is not a limit point of E. for $\varepsilon < \frac{1}{6}$,

$B^{\times}_{\varepsilon}(\frac{1}{2}) \cap E = \phi$.

$\forall \varepsilon > 0$



$E' = \{0\}$.

- **Def**: (closure). $E \subset S$ any subset, the closure of E. is the intersection of all closed subsets containg E,

(E^-)
$$\overline{E} := \bigcap_{\substack{F \subset S \text{ closed} \\ F \supset E}} F.$$

Prop: $\overline{E} = E \cup E'$

- boundary. $\partial E = \overline{E} \setminus E^{\circ}$.

(Ross 13.9)

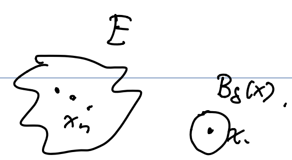
Prop: Let $E \subset (S, d)$.

(1) E is closed $\iff$ $E = \overline{E}$

(2) E is closed $\iff$. $\forall$ convergent sequence $x_n \to x$ in S, if $x_n \in E$ $\forall n$, then $x \in E$.

Pf(2): $\Rightarrow$. suppose E is closed. $x_n \to x$, $x_n \in E$, but $x \notin E$.

then $\exists \delta > 0$, s.t. $B_\delta(x) \cap E = \phi$. $\iff B_\delta(x) \subset E^c$



this contradicts with $x_n \to x$.

$\Leftarrow$ suffice to prove that, if $x \notin E$, then $\exists \delta > 0$, s.t.

$B_\delta(x) \subset E^c$. Suppose it's impossible to find such δ, i.e.

$\forall \delta > 0$, $B_\delta(x) \cap E \neq \phi$, then take δ to run through $\frac{1}{n}$...

let $x_n \in B_{\frac{1}{n}}(x) \cap E$. then $x_n \to x$ as a seq in S.

By assumption, $x \in E$. Hence we have contradiction with $x \notin E$.

#

Compact set

(S, d) metric space.

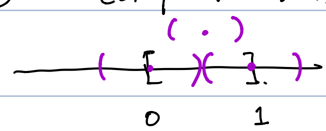
- Def (open cover). Let $E \subset S$. $\{G_\alpha\}$ is a collection of open sets. We say $\{G_\alpha\}$ is an open cover of E, if

$$E \subset \bigcup_\alpha G_\alpha.$$

- Def: $K \subset S$ is a compact subset, if for any open cover of K, there exists a finite subcover. i.e. if $\{G_\alpha\}$ is an open cover, then $\exists$ $\alpha_1, \cdots, \alpha_n$ indices, s.t.

$$K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}.$$

Ex': $K = [0,1]$ is compact subset of $\mathbb{R}$.



Consider open cover of the form $\{B_{\frac{1}{2}}(x) : x \in K\}$.

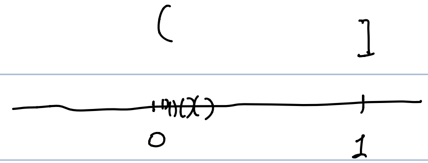
$$K \subset \bigcup_{x \in K} B_{\frac{1}{2}}(x).$$

(not a proof that K is compact).

a finite subcover can be taken as

$$K \subset B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(\tfrac{1}{2}) \cup B_{\frac{1}{2}}(1).$$

Ex 2: $K = (0, 1]$ is not a compact subset.



$G_1 = B_{\frac{1}{2}}(1) = (\frac{1}{2}, \frac{3}{2})$

$G_2 = B_{\frac{1}{4}}(\frac{1}{2}) = (\frac{1}{2} - \frac{1}{4}, \frac{1}{2} + \frac{1}{4})$

$G_3 = B_{\frac{1}{6}}(\frac{1}{3}) = (\cdots)$.

let $G_n = B_{\frac{1}{2n}}(\frac{1}{n})$, $n \in \mathbb{N}$.

then $K \subset \bigcup_{n \in \mathbb{N}} G_n$.

no finite subcover exists. Indeed,

$$G_{n_1} \cup G_{n_2} \cup \cdots \cup \underbrace{G_{n_m}}. \qquad n_1 < n_2 < \cdots < n_m.$$

if $0 < x < \frac{1}{n_m} - \frac{1}{2n_m} = \frac{1}{2n_m}$, then x is not in this union.

Ex 3: $K = \{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\} \cup \{0\}$. $\subset \mathbb{R}$. is compact.

Pf: Let $\{G_\alpha\}$ be any open cover of K.

- $\exists\, G_{\alpha_0}$, s.t. $0 \in G_{\alpha_0}$ $\Rightarrow$ $\exists\, \delta > 0$. s.t. $B_\delta(0) \subset G_{\alpha_0}$.
  Let N be large enough, s.t. $\frac{1}{N} < \delta$, then $\forall n > N$.
  $\frac{1}{n} \in B_\delta(0) \subset G_{\alpha_0}$.
- $\forall n \leq N$, $\exists\, G_{\alpha_n} \ni \frac{1}{n}$. then we have a finite subcover. $G_{\alpha_0} \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_N}$.

covers $\frac{1}{N+1}, \frac{1}{N+2}, \cdots, 0$ cover $\frac{1}{1}$ covers $\frac{1}{N}$.

#

Def (Sequentially compact). $E \subset S$ is seq. compact, if any sequence in E has a convergent subseq in E. (the limit point is also in E).

Thm 1:

- For any metric space (S, d), $E \subset S$.

$$E \text{ compact} \iff E \text{ sequentially compact.}$$

- Consider $\mathbb{R}^n$ with Euclidean metric $d(x,y) = |x-y|$.

Thm 2: (Heine-Borel) $E \subset \mathbb{R}^n$ is compact

$$\overset{\text{easy}}{\Longrightarrow} \quad E \text{ is closed and bounded.}$$
$$\underset{\text{hard, uses } \mathbb{R}^n}{\Longleftarrow}$$

Sketch of proof for Thm 1:

$\Rightarrow$ if E is compact, we need to show that for any sequence x_n in E, there is a convergent subseq. Suppose there is no such convergent subseq, then $\forall y \in E$, $\exists \delta(y) > 0$, s.t. $B_{\delta(y)}(y)$ meet the seq (x_n) finitely many times. Then.

$$E \subset \bigcup_{y \in E} B_{\delta(y)}(y).$$

By compactness of E, we have a finite sub cover. indexed by $y_1, \cdots, y_n$. Then.

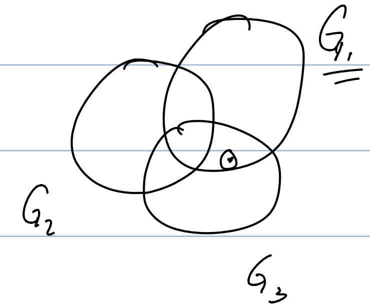
$$E \subset \bigcup_{i=1}^{n} B_{\delta(y_i)}(y_i) \quad \text{meet the seq } (x_n)$$

only finitely many times, hence we have a ~~cod~~ contradiction.

「 $A \subset S$, A meet (x_n) finitely many times means $\{n \mid x_n \in A\}$ is finite. 」

$\Leftarrow$ If E is sequentially compact, then.

Lemma 1: for any open cover $\{G_\alpha\}$ of E, $\exists \delta > 0$, s.t. $\forall x \in E$, $B_\delta(x)$ is contain in some G_α.



Lemma 2: $\forall \delta > 0$, $\exists$ finitely many points $x_1, \cdots, x_N$, s.t.

$$E \subset B_\delta(x_1) \cup \cdots \cup B_\delta(x_N).$$

Given the two Lemma, for any open cover $\{G_\alpha\}$, we first find such δ as in Lemma 1. Then we find a collection of centers $x_1, \cdots, x_N$ as in Lemma 2. Then find $G_{\alpha_i} \supset B_\delta(x_i)$ $\forall i \in \{1, \cdots, N\}$. Then

$$E \subseteq \bigcup_{i=1}^{N} B_\delta(x_i) \subset \bigcup_{i=1}^{N} G_{\alpha_i}.$$