Ross §13. metric space and topology.

midterm 1 :
- closed book. closed notes. (cheat sheet allowed).
- problem : proof / True or false / examples.
- alternative time slot : 9 pm − 10:30 (PT)
  10:00pm − 11:30pm

1. Metric Space. $= (S, d)$

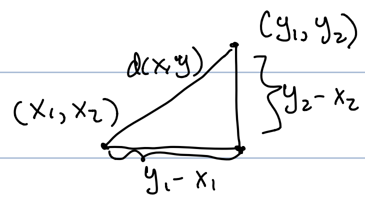
- S : set
- d : $S \times S \to \mathbb{R}_{\geq 0}$

such that :
① $d(x,y) \geq 0$ and $d(x,y) = 0 \iff x = y$.
② $d(x,y) = d(y,x)$
③ $d(x,y) + d(y,z) \geq d(x,z)$.

Ex :
① $S = \mathbb{R}$, $d(x,y) = |x-y|$

② $S = \mathbb{R}^2$, $d(\vec{x},\vec{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

$\vec{x} = (x_1, x_2)$
$\vec{y} = (y_1, y_2)$



Euclidean distance.

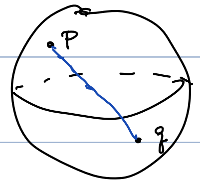
③ $S = \mathbb{R}^n$. $d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

more examples :

- general method : Let (S, d) be a metric space, and let $A \subset S$ a subset (nonempty), then (A, d) is a metric space.

  ↑ restrict from $S \times S$ to $A \times A$.

- Ex : ①, $S = S^2 \subset \mathbb{R}^3$.



$d(p,q)$ = length of the coord connecting p, q.

$= d_{\mathbb{R}^3}(p,q)$

↖ view p, q as points in $\mathbb{R}^3$.

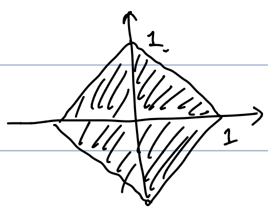
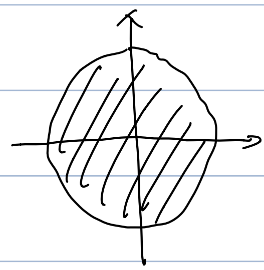
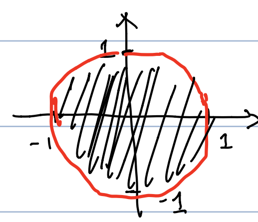
<u>More "exotic" metric:</u>

(1). L^p- distance on $\mathbb{R}^n$, $1 \le p < \infty$.

$$d(x,y) = \left[\sum_{i=1}^n |x_i - y_i|^p\right]^{\frac{1}{p}}.$$

Euclidean distance is L^2- distance.

Ex: what does unit ball in $\mathbb{R}^2$ look like in various distance: $B_{r=1}(0) = \{ x \in \mathbb{R}^2 \mid d(x,0) \le 1 \} = \{ x \in \mathbb{R}^2 \mid |x_1|^p + |x_2|^p \le 1 \}$

$p=1$,	$p=2$,	$p=3$.
$\lvert x_1\rvert + \lvert x_2\rvert \le 1$.		$\lvert x_1\rvert^3 + \lvert x_2\rvert^3 \le 1$
		

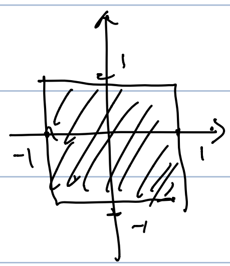
more bulged outward.

$p \to \infty$.

$$d_{\max}(x,y) = d_\infty(x,y) = \lim_{p\to\infty} d_p(x,y) = \max\{ |x_i - y_i| \mid i = 1, \dots, n \}$$

↖ L^p- distance

$p \to \infty$



• Let (S,d) be a metric space, let $(s_n)_n$ be a sequence in S. Then

① $(s_n)_n$ is Cauchy, if $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n, m > N$, $d(s_n, s_m) < \varepsilon$.

② $(s_n)_n$ converges to $s \in S$, if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n > N$,

$$d(s_n, s) < \varepsilon.$$

Lemma: if $(s_n)_n$ is convergent to s, then s_n is Cauchy.

Pf: $\forall \varepsilon > 0$, $\exists N > 0$, s.t.

$$d(s_n, s) < \frac{\varepsilon}{2} \qquad \forall n > N.$$

Thus $\forall n, m > N$, we have

$$d(s_n, s_m) \leq d(s_n, s) + d(s_m, s) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \#.$$

Def: A metric space (S, d) is called complete, if every Cauchy sequence has a limit in S.

- Example of non-complete metric space:

① $S = \mathbb{R} \setminus \{0\}$.

② $S = \mathbb{Q}$

How to take "completion" of a metric space?

idea: • define $\bar{S}$ as the set of Cauchy sequences, modulo equivalence relations.

$$\bar{S} = \{ (S_n)_n \mid (S_n)_n \text{ Cauchy} \} / \sim$$

$(S_n) \sim (t_n)$, if a new seq $C_n = \begin{cases} S_k & n = 2k \\ t_k & n = 2k+1 \end{cases}$

is Cauchy. , or equivalently, $\forall \varepsilon > 0$, $\exists N > 0$.

s.t. $\forall n > N$. $d(S_n, t_n) < \varepsilon$.

Let $[(S_n)_n]$ denote the equivalence class of the Cauchy seq $(S_n)_n$. then.

① $S \hookrightarrow \bar{S}$, by forming constant seq.

② $\lim_{n \to \infty} S_n = [(S_n)]_n$, if (S_n) is Cauchy.

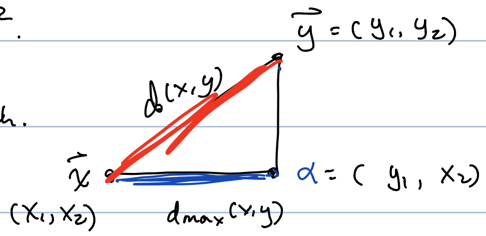
• Now, we consider $(S = \mathbb{R}^n, d = \text{Euclidean distance})$.

Lemma: $\forall x, y \in \mathbb{R}^n$.

(a) $d(x,y) \le n \cdot \underbrace{\max \{ |x_i - y_i| \mid i = 1, \dots, n \}}_{d_{max}(x,y)}$

(b) $d_{max}(x,y) \le \quad \cdot \, d(x,y)$

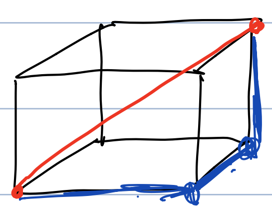
Ex: $\mathbb{R}^2$. x, y as such.



$\alpha = (y_2, x_1)$.

$$\begin{aligned} d(x,y) &\le d(x,\alpha) + d(\alpha,y) \\ &\le d_{max}(x,y) + d_{max}(x,y) \\ &= 2 \cdot d_{max}(x,y). \end{aligned}$$

$\mathbb{R}^3$:



$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \ge \sqrt{(x_{i_0} - y_{i_0})^2} = |x_{i_0} - y_{i_0}|$$

for any i_0.

hence $d(x,y) \geq \max_{i_0} |x_{i_0} - y_{i_0}| = d_{max}(x,y)$.

$\Rightarrow$ Exist constants $C_1, C_2 > 0$, such that

$$C_1 \cdot d_{max}(x,y) < d(x,y) < C_2 \cdot d_{max}(x,y)$$

Actually, $\forall p \geq 1$ $\exists$ $C_1, C_2 > 0$. s.t.

$$C_1 \cdot d_p(x,y) < d(x,y) < C_2 \cdot d_p(x,y).$$ #

Lemma: Let $(S_n)_{n\in\mathbb{N}}$ be a seq in $\mathbb{R}^m$.

$S_n = (S_n^{(1)}, \cdots, S_n^{(m)}) \in \mathbb{R}^m$. Then.

(1) $(S_n)_n$ is Cauchy $\Leftrightarrow$ $\forall i \in \{1,\cdots,m\}$. $(S_n^{(i)})_n$ is Cauchy in $\mathbb{R}$

(2) $(S_n)_n$ is convergent $\Leftrightarrow$ $(S_n^{(i)})_n$ is convergent, $\forall i \in \{1,\cdots,m\}$.

Pf: (1). $\Rightarrow$ if (S_n) is Cauchy, then $\forall \varepsilon > 0$, $\exists N > 0$, s.t. $\forall n_1, n_2 > N$.

$$d(S_{n_1}, S_{n_2}) < \varepsilon.$$

But $d(S_{n_1}, S_{n_2}) \geq d(S_{n_1}^{(i)}, S_{n_2}^{(i)})$ $\forall i \in \{1,\cdots,m\}$

thus, $(S_n^{(i)})_n$ is Cauchy.

$\Leftarrow$ if $(S_n^{(i)})$ is Cauchy, $\forall i \in \{1,\cdots,m\}$ Then.

$\forall \varepsilon > 0$, exists $N^{(i)} > 0$, s.t. $\forall n_1, n_2 > N^{(i)}$,

$$d(S_{n_1}^{(i)}, S_{n_2}^{(i)}) < \varepsilon.$$

Hence, if we take $N = \max\{N^{(1)}, \cdots, N^{(m)}\}$,

then $\forall\, n_1, n_2 > N$,

$$d(S_{n_1}^{(i)}, S_{n_2}^{(i)}) < \varepsilon \qquad \forall\, i \in \{1, \cdots, m\}.$$

$$\Rightarrow \quad d_{max}(S_{n_1}, S_{n_2}) < \varepsilon.$$

$$\Rightarrow \quad d(S_{n_1}, S_{n_2}) \leq n \cdot d_{max} < n \cdot \varepsilon.$$

Hence, we have shown $(S_n)_n$ is Cauchy.

Thm: $\mathbb{R}^n$ is a complete metric space.

Pf: Let (S_n) be a Cauchy seq in $\mathbb{R}^n$

by Lemma. $\forall\, i \in \{1, \cdots, m\}$,

$\Rightarrow$ $(S_n^{(i)})_n$ is a Cauchy

by completeness in $\mathbb{R}$

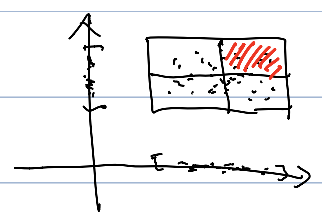
$\Rightarrow$ $\forall\, i$, $(S_n^{(i)})_n$ is convergent

by Lemma.

$\Rightarrow$ $(S_n)_n$ is convergent. #.

Thm: (Bolzano-Weierstrass) Every bounded sequence in $\mathbb{R}^m$ has a convergent sub seq.

$\because$ $(S_n^{(1)})_n$ is a bounded seq in $\mathbb{R}$.



Pf: (sketch): (1) take a subseq, such that $(S_{n_k}^{(1)})_{n_k}$ is convergent, the we rename the subseq as the original seq.

(2) take a further subseq, such that $(S_{n_k}^{(2)})_k$ is convergent,

$\vdots$

after we do it m times, we get a subseq. that is convergent in all coordinate component. #.

三

<u>Topology</u>: A topology on a set S, is a collection of subsets, which we call open subsets, such that

① S, ϕ are open

② if $\{U_i\}_{i \in I}$ is a collection of open subsets, then $\bigcup_{i \in I} U_i$ is open.

③ if $\{U_i\}_{i=1}^{N}$ is a finite collection of open, then $\bigcap_{i=1}^{N} U_i$ is open.

<u>Topology for metric space</u>: Let (S, d) be a metric space. $\forall r > 0$, $p \in S$, we declare these

$$\underline{B_r(p)} = \{ x \in S \mid d(p, x) < r \}.$$

to be open. Then, the minimal collection of subsets of S satisfying the axioms ①, ②, ③ forms a topology induced by d.

<u>Direct way</u> to define open set: for (S, d).

- We say $U \subset S$ is <u>open</u>, if $\forall p \in U$, $\exists r > 0$, s.t. $B_r(p) \subset U$.

$$U = \bigcup_{p \in U} B_{r(p)}(p)$$