Today: · §12. liminf. limsup.

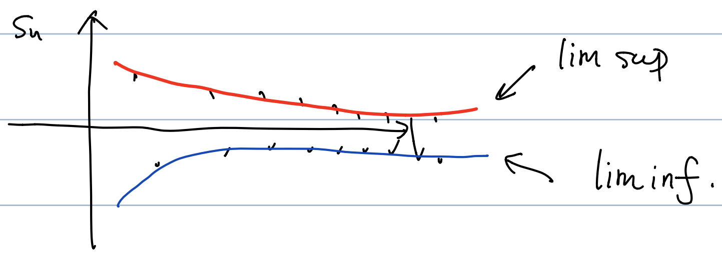
· review for midterm

next Tue: §13. metric space and topology.

· $\limsup (s_n) = \lim_{N\to\infty} \left(\sup_{n\geq N} (s_n) \right)$

$\liminf (s_n) = \lim_{N\to\infty} \left(\inf_{n\geq N} (s_n) \right).$

Ex:



· $\liminf (s_n) \leq \limsup (s_n).$

Pf: $\sup_{n\geq N} (s_n) \geq \inf_{n\geq N} (s_n).$, then take limit $N\to\infty$.

· If (s_{n_k}) is a subseq, then.

$$\limsup (s_{n_k}) \leq \limsup (s_n).$$

Pf: $\limsup (s_n) = \lim_{N\to\infty} A_N$ $\qquad A_N = \sup_{n\geq N} (s_n).$

$= \lim_{k\to\infty} A_{n_k}$

$\limsup_k (s_{n_k}) = \lim_{k\to\infty} A'_{N_k}$ $\qquad A'_{N_k} = \sup_{l\geq k} (s_{n_l})$

since the set $\{n_l \mid l\geq k\} \subset \{n \mid n\geq n_k\}$

hence, $\sup \{s_{n_l} \mid l\geq k\} \leq \sup(s_n \mid n\geq n_k).$

$\Rightarrow \quad A'_{n_k} \leq A_{n_k}$

$$\Rightarrow \underbrace{\lim_k A'_{n_k}}_{\limsup_k (s_{n_k})} \le \lim_k A_{n_k} = \limsup(s_n)$$

#.

- If $\limsup(s_n) = \liminf(s_n)$, then $\lim s_n$ exists.

<u>Thm</u> 12.1. Let s_n be a sequence. s.t. $\lim s_n = s > 0$. ($\in \mathbb{R}$)

- Let t_n be a bound sequence, $\beta = \limsup t_n$.

Then. $\limsup(s_n \cdot t_n) = s \cdot \limsup(t_n)$.

<u>Observation</u>: ① recall: if t_n also converge, say to β.

then $\lim(s_n \cdot t_n) = \lim(s_n) \cdot \lim(t_n) = s \cdot \beta$.

② what are the possible sub sequential limits of $(s_n \cdot t_n)$?

- One way to produce convergent sub seq of $(s_n \cdot t_n)$ is that, pick a convergent subseq (t_{n_k}) in t_n, then $(s_{n_k} \cdot t_{n_k})_k$ is convergent.
- Q: if $(s_{n_k} \cdot t_{n_k})$ converge, does that mean (t_{n_k}) converge? ✓

[if $\lim a_n$ exist, $\lim b_n \neq 0$, ($b_n \neq 0$) then $\lim\left(\frac{a_n}{b_n}\right)$ exist,

$$= \frac{\lim(a_n)}{\lim(b_n)}$$

$$\underbrace{\text{Sub Seq Lim of } (s_n \cdot t_n)}_{A} = s \cdot \underbrace{\text{sub seq lim } (t_n)}_{B}.$$

$A = s \cdot B.$

Then $\sup(A) = s \cdot \sup(B)$

$$\Rightarrow \quad \limsup(s_n \cdot t_n) = s \cdot \limsup(t_n). \qquad \#_1.$$

<u>Pf #2</u>: (using epsilon of ~~roo~~ room trick).

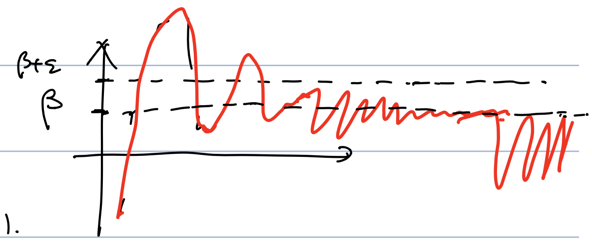
- To prove $\limsup(s_n \cdot t_n) \le s \cdot \limsup(t_n)$,

  we just need to prove

$$\limsup(s_n \cdot t_n) \le s \cdot \limsup(t_n) + \varepsilon. \quad \forall \varepsilon > 0.$$

- Let $\beta = \limsup(t_n)$. $\forall \varepsilon > 0$, $\exists N > 0$. s.t.

$$t_n < \beta + \varepsilon \qquad \forall n > N.$$



- (a) $\beta > 0$. $\forall \varepsilon_1 > 0$, $\exists N_1 > 0$,

  s.t. $|s_n - s| < \underbrace{s \cdot \varepsilon_1}$, $\forall n > N_1$.

$$1 - \varepsilon_1 < (s_n / s) < 1 + \varepsilon_1 \quad \Leftarrow \quad s\varepsilon_1 < s_n - s < s \cdot \varepsilon_1$$

$\forall n > N_1$ and $n \ge N$.

$$s_n \cdot t_n < s(1+\varepsilon_1) \cdot (\beta + \varepsilon).$$

$0 < a < b$, $c < d$, $d > 0$. $\Rightarrow a \cdot c < bd$

$$\Rightarrow \limsup(s_n t_n) < s(1+\varepsilon_1) \cdot (\beta + \varepsilon). \quad \forall \varepsilon > 0, \ \varepsilon_1 > 0.$$

$$\Rightarrow \limsup(s_n \cdot t_n) \le s \cdot \beta. \quad \left(\text{taking limit of } \varepsilon \to 0, \ \varepsilon_1 \to 0\right)$$

- $\beta = 0$, $\beta < 0$ for exercise.

Together, they prove one direction

$$\limsup(s_n \cdot t_n) \le s \cdot \limsup(t_n). \qquad (1)$$

To get the other direction, we do.

$$\limsup(t_n) = \limsup\left(\frac{1}{s_n} \cdot s_n t_n\right).$$

$$\overset{\text{by (1)}}{\le} \lim\left(\frac{1}{s_n}\right) \cdot \limsup(s_n t_n).$$

$$= \frac{1}{s} \cdot \limsup (s_n t_n)$$

$$\Rightarrow \quad s \cdot \limsup (t_n) \leq \limsup (s_n t_n) \qquad ②.$$

$$① + ② \quad \Rightarrow \quad s \cdot \limsup t_n = \limsup (s_n \cdot t_n). \qquad \#.$$

Thm 12.2 Let s_n be a sequence of positive numbers.

then

$$\liminf \left(\frac{s_{n+1}}{s_n}\right) \underset{①}{\leq} \liminf (s_n)^{\frac{1}{n}} \underset{②}{\leq} \limsup (s_n)^{\frac{1}{n}} \underset{③}{\leq} \limsup \left(\frac{s_{n+1}}{s_n}\right).$$

Observation: • $\frac{s_{n+1}}{s_n}$ sequence of ratio. , it measure changes of nearby elements.

• $(s_n)^{\frac{1}{n}}$. • Recall, if $a > 0$, then. $\lim_{n \to \infty} (a)^{\frac{1}{n}} = 1.$

• Hence, if $0 < a < s_n < b$, then.

$$a^{\frac{1}{n}} < (s_n)^{\frac{1}{n}} < b^{\frac{1}{n}}$$

$$\Rightarrow \quad 1 = \lim (a^{\frac{1}{n}}) \leq \limsup (s_n)^{\frac{1}{n}} \leq \lim b^{\frac{1}{n}} = 1$$

$\liminf (s_n)^{\frac{1}{n}}$.

if $0 < a < s_n < b, \forall n$

$\Rightarrow$ $\lim (s_n)^{\frac{1}{n}}$ exist , and $= 1$.

$\boxed{\text{best case}}$.

• $(s_n)^{\frac{1}{n}} = q$, $\qquad \frac{s_{n+1}}{s_n} = q$. $\qquad (q > 0)$.

$\Rightarrow$ $s_n = q^n$. $\qquad (s_n) = 1, q, q^2, \cdots$

• slight variation of the best case :

$s_n = a \cdot q^n$. $\qquad a > 0, q > 0$.

$$\Rightarrow \begin{cases} \dfrac{S_{n+1}}{S_n} = \dfrac{a\cdot q^{n+1}}{a\cdot q^n} = q. \\ \\ (S_n)^{\frac{1}{n}} = \underline{\underline{a^{\frac{1}{n}}}}\cdot q \end{cases} \qquad \text{but } a^{\frac{1}{n}} \to 1.$$

$$\Rightarrow \lim (S_n)^{\frac{1}{n}} = q \qquad \text{(as before)}$$

(worst case)

- Ex: $S_n = \begin{cases} 2 & n \text{ is even} \\ 1 & n \text{ is odd.} \end{cases}$

then. $(S_n) = 1, 2, 1, 2, 1, 2,$

$(S_{n+1}/S_n) = 2, \frac{1}{2}, 2, \frac{1}{2}, \cdots$

$(S_n)^{\frac{1}{n}} = 1, 2^{\frac{1}{2}}, 1, 2^{\frac{1}{4}}, \cdots$, $\lim (S_n)^{\frac{1}{n}} = 1$
($\because S_n$ is bounded)

So, $\limsup (S_{n+1}/S_n) = 2$

$\liminf (S_{n+1}/S_n) = 1/2.$

$\lim (S_n)^{\frac{1}{n}} = 1,$

thm claims:

$\frac{1}{2} \leq 1 \leq 1 \leq 2$ ✓.

Pf: Proof for ③,

$$\underline{\limsup (S_n)^{\frac{1}{n}}} \leq \limsup \left(\frac{S_{n+1}}{S_n}\right).$$

If $\limsup\left(\frac{S_{n+1}}{S_n}\right) = +\infty$, then there is nothing to prove.

Since S_{n+1}/S_n are positive, hence $\limsup (S_{n+1}/S_n)$ cannot be $-\infty$.

Hence, we only need to consider the case that

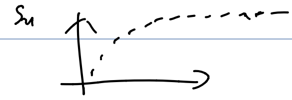
$$L = \limsup\left(\frac{S_{n+1}}{S_n}\right) \in \mathbb{R}.$$

(in particular, we know $L \geq 0$).

Quiz: if S_n is a <u>bounded</u> positive seq, is it true that $\frac{S_{n+1}}{S_n}$ is a bounded seq? (Ans: No).

(if $0 < a < 1$, $0 < b < 1$, do we have upper bound for a/b ?
take $a = 1/2$, $b = \frac{1}{n}$, then $a/b = \frac{n}{2}$.)

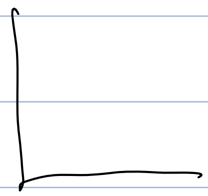
- if S_n is monotone., then S_{n+1}/S_n can not be getting bigger

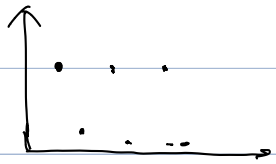


Pf: S_n bounded and monotone $\nearrow$ $\Rightarrow$ $S_n \to S > 0$

$$\Rightarrow \lim \frac{S_{n+1}}{S_n} = \frac{\lim S_{n+1}}{\lim S_n} = \frac{S}{S} = 1. \quad \#.$$

- How to get S_n bounded, but S_{n+1}/S_n unbounded ?





$$S_n = \begin{cases} 1 & n \text{ even} \\ 1/n & n \text{ odd.} \end{cases}$$

Quiz: if S_n is positive and bounded, is it possible that $S_{n+1}/S_n \to 0$?

Ans: Yes. $S_n = \frac{1}{n!}$, $S_{n+1}/S_n = \frac{1}{n+1}$.

To prove $\limsup (S_n)^{\frac{1}{n}} \le L.$, only need to show, $\forall \varepsilon > 0$,

$$\limsup (S_n)^{\frac{1}{n}} \le L + \varepsilon. \quad (*)$$

For this ε, we can choose $N > 0$, integer. s.t. $\forall n > N$,

$$\left(\frac{S_{n+1}}{S_n}\right) \le L + \varepsilon.$$

$$\Rightarrow \frac{S_{N+1}}{S_N} \le L + \varepsilon, \quad \frac{S_{N+2}}{S_{N+1}} \le L + \varepsilon, \cdots$$

$$\Rightarrow \quad \frac{S_{N+k}}{S_N} \leqslant (L+\varepsilon)^k.$$

$$\Rightarrow \quad S_{N+k} \leqslant S_N \cdot (L+\varepsilon)^k.$$

$$\Rightarrow \quad \forall n \geqslant N, \qquad S_n^{\frac{1}{n}} \leqslant \left[S_N \cdot (L+\varepsilon)^{n-N} \right]^{\frac{1}{n}}.$$

$$= \left(\frac{S_N}{(L+\varepsilon)^N} \right)^{\frac{1}{n}} \cdot (L+\varepsilon).$$

$$\Rightarrow \quad \limsup \, (S_n)^{\frac{1}{n}} \leqslant \limsup \, \underbrace{\left(\frac{S_N}{(L+\varepsilon)^N} \right)^{\frac{1}{n}}} \cdot \underbrace{(L+\varepsilon)}$$

$$= (L+\varepsilon) \cdot \limsup_{n\to\infty} \left(\frac{S_N}{(L+\varepsilon)^N} \right)^{\frac{1}{n}}$$

$$= (L+\varepsilon).$$

Hence $(*)$ is true, $\forall \varepsilon > 0$. Thus ③ holds.