Today: ① §12. lim sup and lim inf

② review.

Recall: • Let (s_n) be any seq of real number

$$\limsup_n s_n = \lim_{N\to\infty}\left(\sup_{n>N} s_n\right), \quad \liminf s_n = \lim_{N\to\infty}\left(\inf_{n>N} s_n\right).$$

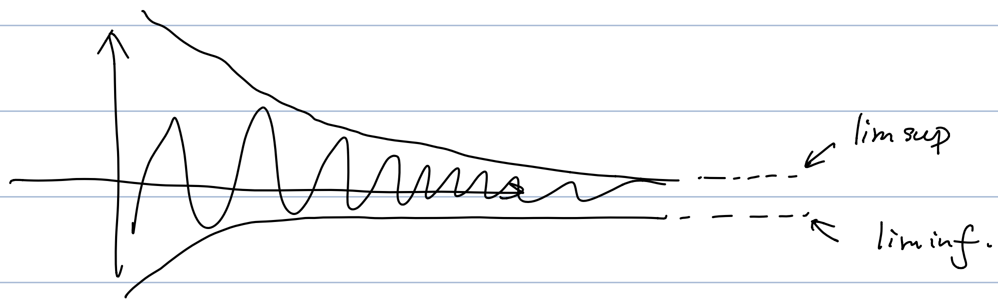
• They only care about the "tail behavior",

• $\forall \varepsilon>0$, $\exists N>0$, s.t.

$$(\limsup s_n)+\varepsilon > s_n \qquad \forall n>N.$$

similarly $$(\liminf s_n)-\varepsilon < s_n \qquad \forall n>N$$

• Picture:



• In general, $\liminf(s_n) \leq \limsup(s_n)$.

If they are equal, then $\lim s_n$ exists.

Thm: Let s_n be a seq, with limit $s>0$.

Let t_n be a seq, bounded.

Then $$\limsup(s_n \cdot t_n) = s \cdot \limsup t_n.$$

Observation: ① simple case. s_n is a constant seq.

then $$\limsup(s\cdot t_n) = s\cdot\limsup t_n.$$

② In general, we have "noise" ~~off~~ in S_n.

$$S_n = \underline{\underline{S}} + \underbrace{(S_n - S)}_{= \text{ noise.}}$$ will be smaller and smaller as $n \to \infty$.

③ $\beta = \limsup t_n$, exists in $\mathbb{R}$. maybe $+, -,$ or 0.

One simplification is that, by replacing S_n with S_n/S. we may assume $S_n \to 1$.

Pf: WLOG, assume $S_n \to 1$.

Try to prove one direction first:

$$\limsup (S_n \cdot t_n) \geq \limsup t_n = \beta$$

There exists a subseq (t_{n_k}), that converge to β. Then, with the same index subset (n_k), we consider (S_{n_k}) subseq in (S_n). Since $\lim S_n = 1$, hence $\lim_k S_{n_k} = 1$.

$$\lim_{k\to\infty} (S_{n_k} \cdot t_{n_k}) = (\lim_k S_{n_k})(\lim_k t_{n_k}) = 1 \cdot \beta = \beta.$$

$(S_{n_k} \cdot t_{n_k})_{k\in\mathbb{N}}$ is a subseq of $(S_n t_n)_{n\in\mathbb{N}}$

$\because$ $\limsup (S_n t_n)$ is the largest among all possible subsequential limits of $(S_n t_n)$, hence

$$\limsup (S_n t_n) \geq \beta.$$

To get the other direction, we consider

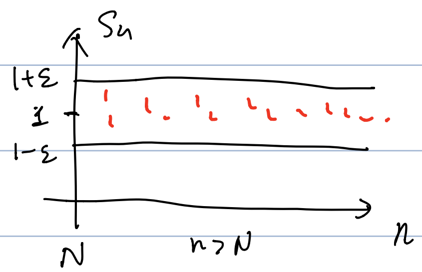
$\limsup t_n = \limsup \left(\frac{1}{s_n} \cdot s_n t_n\right).$

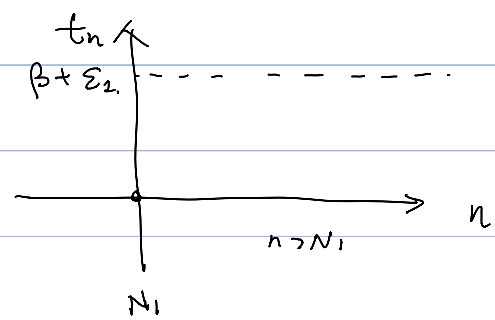
$\because \lim \frac{1}{s_n} = \frac{1}{\lim s_n} = 1.$,

$\therefore \limsup t_n \geq \lim\left(\frac{1}{s_n}\right) \cdot \limsup(s_n t_n) = \limsup (s_n t_n).$

$\Rightarrow \limsup t_n = \limsup (s_n t_n)$ #.

<u>sketch of pf (#2)</u>: ① $\because s_n \to 1.$ $\therefore \forall \varepsilon > 0, \exists N > 0, \forall n > N,$
$1-\varepsilon < s_n < 1+\varepsilon$. ② $\limsup t_n = \beta$., $\forall \varepsilon_1 > 0, \exists N_1 > 0,$
s.t. $\forall n > N_1$, $t_n < \beta + \varepsilon_1$.





If $\beta = 0$, then $s \cdot \limsup t_n = s \cdot 0 = 0.$
one ~~has~~ only need prove $\limsup (s_n \cdot t_n) = 0.$

If $\beta \neq 0$, then ~~...~~ take $0 < \varepsilon_1 < |\beta|/2$. so that
$\beta + \varepsilon_1$ has same sign as β.

Consider the case $\beta \neq 0$ first.

- $\beta > 0$. $\because$ $\forall n > \max(N, N_1)$,

$s_n < 1+\varepsilon.$

$t_n < \beta + \varepsilon_1.$

$\Rightarrow s_n \cdot t_n < (1+\varepsilon)(\beta+\varepsilon_1)$ $\forall n > \max(N, N_1)$.

$$\Rightarrow \quad \limsup (s_n t_n) < (1+\varepsilon)(\beta+\varepsilon_1).$$

since this is true $\forall \varepsilon, \varepsilon_1$, $|\varepsilon_1| < |\beta|/2$.

$$\Rightarrow \quad \limsup (s_n t_n) \leq \beta.$$

- $\beta < 0$, then. $\forall\ |\varepsilon| < 1/2,\ |\varepsilon_1| < |\beta|/2$, positive.

$$s_n > 1-\varepsilon > 0.$$

$$t_n < \beta+\varepsilon_1 < 0$$

$$\Rightarrow \quad s_n \cdot t_n < (\beta+\varepsilon_1)(1-\varepsilon) < 0 \qquad \forall n > \max(N, N_1)$$

$$\Rightarrow \quad \limsup (s_n t_n) \leq \beta \cdot 1$$

- $\beta = 0$.

$$s_n < 1+\varepsilon. \qquad \forall n \text{ large.}$$

$$t_n < \varepsilon_1$$

$$\Rightarrow \quad s_n \cdot t_n < (1+\varepsilon)\varepsilon_1.$$

$$\Rightarrow \quad \limsup s_n t_n < (1+\varepsilon)\varepsilon_1 \qquad \forall \varepsilon, \varepsilon_1 > 0. \text{ small.}$$

$$\Rightarrow \quad \limsup s_n t_n \leq 0.$$

To get lower bound, one use the same method as proof #1.

#.

Thm 12.2: Let (s_n) be a sequence of positive numbers.

Then we have.

$$\liminf\left(\frac{s_{n+1}}{s_n}\right) \underset{①}{\leq} \liminf (s_n)^{\frac{1}{n}} \underset{②}{\leq} \limsup (s_n)^{1/n} \underset{③}{\leq} \limsup \left(\frac{s_{n+1}}{s_n}\right)$$

Observation: (1) consider the case, $s_n = q^n$ $(q > 0)$.

then $(s_n)^{\frac{1}{n}} = q.$ $\quad s_{n+1}/s_n = q.$

(2). if $S_n = q^n \cdot A$, $A>0$. then

$S_{n+1}/S_n = q$.

$(S_n)^{\frac{1}{n}} = \underbrace{(A^{\frac{1}{n}})} \cdot q$.

we have $\lim_{n\to\infty} A^{\frac{1}{n}} = A^0 = 1$. (see §7. or §9 results).

$\lim (S_n)^{\frac{1}{n}} = q$.

(3) if $S_n = \begin{cases} 2 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd.} \end{cases}$

then $S_{n+1}/S_n = \begin{cases} 2/1 & n \text{ odd} \\ \text{or } 1/2. & n \text{ even.} \end{cases}$

Thus. $\limsup (S_{n+1}/S_n) = 2$ ✓

$\liminf (S_{n+1}/S_n) = 1/2$. ✓

How about $(S_n)^{\frac{1}{n}}$? $\because$ $\underline{1^{\frac{1}{n}} \leq (S_n)^{\frac{1}{n}} \leq 2^{\frac{1}{n}}}$

and $\lim 2^{\frac{1}{n}} = 1$, $\lim 1^{\frac{1}{n}} = 1$, $\therefore$ $\lim (S_n)^{\frac{1}{n}} = 1$.

To prove ③

Pf: Let $L = \limsup (S_{n+1}/S_n)$. If $L = +\infty$, then there is nothing to prove.

$\because$ $S_{n+1}/S_n > 0$ $\therefore$ $L \geq 0$. Hence, we only need consider $L \in \mathbb{R}$.

To prove $\limsup (S_n)^{\frac{1}{n}} \leq \limsup (S_{n+1}/S_n)$,

only need to prove

$$\limsup (S_n)^{\frac{1}{n}} \leq L+\varepsilon. \quad \forall \varepsilon>0. \quad \underline{\underline{(*)}}$$

$\forall \varepsilon>0$, $\exists N>0$ (integer), s.t. $\forall n \geq N$, $(S_{n+1}/S_n) < L+\varepsilon$.

$\Rightarrow$ $\frac{S_{N+1}}{S_N} < L+\varepsilon$, $\frac{S_{N+2}}{S_{N+1}} < L+\varepsilon$, ----

$\Rightarrow$ $S_{N+k} < S_N \cdot (L+\varepsilon)^k = \frac{S_N}{(L+\varepsilon)^N} (L+\varepsilon)^{N+k}$

Thus $\quad (S_{N+k})^{\frac{1}{N+k}} < \left[\frac{S_N}{(L+\varepsilon)^N}\right]^{\frac{1}{N+k}} \cdot (L+\varepsilon) \quad \forall k>0$

$$\Rightarrow \limsup_{k\to\infty} (S_{N+k})^{\frac{1}{N+k}} \leq \limsup_{k} \left(\frac{S_N}{(L+\varepsilon)^N}\right)^{\frac{1}{N+k}} \cdot (L+\varepsilon).$$

$$\Rightarrow \limsup_{n\to\infty} (S_n)^{\frac{1}{n}} \leq \left[\lim_{k\to\infty} \left(\frac{S_N}{(L+\varepsilon)^N}\right)^{\frac{1}{N+k}}\right] \cdot (L+\varepsilon)$$

$$= 1 \cdot (L+\varepsilon).$$

this proves $(*)$, hence inequality ③.

Review: ① real number. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

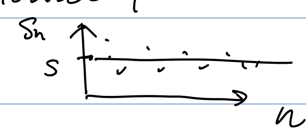
Completeness Axiom: if $S \subset \mathbb{R}$ is bounded, then $\sup(S)$ exists in $\mathbb{R}$.

- difference between $\max(S)$, $\min(S)$ v.s. $\sup(S)$? $\inf(S)$?

they may not exist, if they do exists, they need to be inside S.

② sequence and limit.

- ε-N language
- pictorial presentation



- algebraic operation to limits.

if s_n, t_n converge to s, t,

$\lim (s_n+t_n) = \lim s_n + \lim t_n$, ... etc.

- example limits:
  - if $a>0$, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$.
  - $\lim_{n\to\infty} \frac{1}{n^p} = 0$ if $p>0$.

- comparison between limits:

  if $s_n \le t_n$, and $\lim s_n$, $\lim t_n$ exists in $\mathbb{R}$,

  then $\lim s_n \le \lim t_n$.

- Squeeze Lemma: given (a_n), (b_n), (c_n).

  if $a_n \le b_n \le c_n$ $\forall n$, and $\lim a_n = \lim c_n$.

  then $\lim b_n$ exists.

- Monotone Sequence + Boundness $\Rightarrow$ convergence.

  Cauchy $\Leftrightarrow$ convergence.

- If (s_n) does not converge, we try to extra subseq that converges.

- t is a subseq limit, $\Leftrightarrow$ $\forall \varepsilon > 0$, $\{n \mid |s_n - t| < \varepsilon\}$ is an infinite set.

  (know the proof of construction of subseq).