Recall:

From Tuesday's lecture

Thm (Rolle): Let $f: [a,b] \to \mathbb{R}$ continuous, and $f|_{(a,b)}$ be differentiable. Then, if $f(a) = f(b)$, we have $c \in (a,b)$ such that $f'(c) = 0$.

Pf: If $f(x)$ is a constant function, then for any $c \in (a,b)$, $f'(c) = 0$.

If $f(x)$ is not a constant fcn, $\exists\ t_0 \in (a,b)$, s.t. $f(t_0) \neq f(a)$. WLOG, assume $f(t_0) > f(a)$.

Then, let $x_0 \in [a,b]$ be the point that realizes $\max\{f(x) \mid x \in [a,b]\}$. This is realizable, since $[a,b]$ is compact and f is continuous. Then $f(x_0) \geq f(t_0) > f(a)$, hence $x_0 \neq a$, and $x_0 \neq b$, i.e. $x_0 \in (a,b)$. By Lemma (last time), since x_0 is a ~~local~~ global maxima (hence a local maxima) of f, hence $f'(x_0) = 0$. Take $c = x_0$. #.

Thm (Generalized Mean Value Thm): Let $f, g: [a,b] \to \mathbb{R}$ be cont. fcn, differentiable on (a,b). Then $\exists\ c \in (a,b)$, such that

$$(*) \quad [f(a) - f(b)] \cdot g'(c) = [g(a) - g(b)] \cdot f'(c).$$

$$\Leftrightarrow \quad [f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c).$$

Pf: Let $h(x) = [f(a) - f(b)][g(x) - g(a)] - [f(x) - f(a)][g(a) - g(b)]$

Then $h(a) = [f(a) - f(b)][g(a) - g(a)] - [f(a) - f(a)][g(a) - g(b)]$
$= 0$

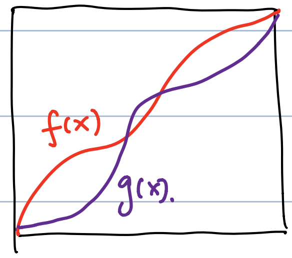
$h(b) = [f(a) - f(b)][g(b) - g(a)] - [f(b) - f(a)][g(a) - g(b)]$
$= 0$

So we may apply Rolle Thm to $h(x)$. , hence there exists a $c \in (a,b)$, such that $h'(c) = 0$.

$\because$ $h'(x) = [f(a) - f(b)] \cdot g'(x) - f'(x)[g(a) - g(b)]$.

$\therefore$ $h'(c) = 0$ implies the desired equality $(*)$. #.

In the special case that $g(a) = f(a)$, $g(b) = f(b)$,

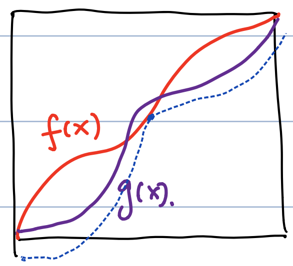


then we are looking for a point $c \in (a,b)$.

s.t. $f'(c) = g'(c)$.

This c will be achieved, if $h(x) = f(x) - g(x)$ has a local max or local min.

To see where $f'(c) = g'(c)$, we may shift the graph of g (up and down), to see the "point of tangency".



Thm : Let $f: [a,b] \to \mathbb{R}$ be continuous, and differentiable over (a,b). Then $\exists$ $c \in (a,b)$, such that

$$[f(b) - f(a)] = (b-a) \cdot f'(c).$$

Pf : Apply the generalized M.V.T., with $g(x) = x$. #

Rmk: mean value theorem relates slope at a point to the difference of the values of the function.

Cor: Suppose $f: [a,b] \to \mathbb{R}$ cont. $f'(x)$ exist for $x \in (a,b)$, and $|f'(x)| \leq M$ for some constant M. Then f is uniformly continuous.

Pf: $\forall \varepsilon > 0$, we may take $\delta = \frac{\varepsilon}{M}$. Then, $\forall$ $x, y \in [a,b]$, $x<y$

we may apply M.V.T for the ~~inte~~ interval (x,y), and get

$$f(y) - f(x) = (y-x) \cdot f'(z). \quad \text{for some } z \in (x,y).$$

$\Rightarrow$

$$|f(y) - f(x)| = |y-x| \cdot |f'(z)| \leq \delta \cdot M \leq \varepsilon.$$

Cor: Let $f: [a,b] \to \mathbb{R}$ cont, and diff over (a,b).

If $f'(x) \geq 0$ $\forall x \in (a,b)$, then f is monotone increasing (i.e. if $y > x$, then $f(y) \geq f(x)$).

If $f'(x) > 0$ $\forall x \in (a,b)$, then f is strictly increasing.

Pf: exercise.

☆ Intermediate Value Theorem for derivatives.

Thm: Assume $f(x)$ is differentiable over $[a,b]$, with $f'(a) < f'(b)$. Then, for each $\mu \in (f'(a), f'(b))$, there exists a $c \in (a,b)$, s.t. $f'(c) = \mu$.

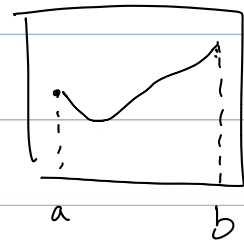
Rmk: This does not follow from IVT for $f'(x)$, since

$f'(x)$ may not be continuous.

Pf: Consider $g(x) = f(x) - \mu \cdot x$, then.

$g'(a) = f'(a) - \mu < 0$, $\quad g'(b) = f'(b) - \mu > 0$.

Hence, a, b are not local minimum of $g(x)$. (hence are not global mimima).



Let c be ~~the~~ a global minimum of $g(x)$ over $[a, b]$, then $c \in (a, b)$., and $g'(c) = 0$.

$\Rightarrow f'(c) = \mu$. #.

L'Hopital Rule.

Ex:

(1) $\lim_{x \to 0} \frac{\sin x}{x}$. $\quad \lim_{x\to 0} \sin x = 0$, $\quad \lim_{x \to 0} x = 0$.

we call it the "$\frac{0}{0}$" type limit. The L'Hopital rules say,

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{(\sin x)'}{x'} = \lim_{x \to 0} \frac{(\cos x)}{1} = \frac{\cos(0)}{1} = 1.$$

(2). $\lim_{x \to \infty} \frac{\log x}{x}$, both numerator and denominator $\to \infty$

"$\frac{\infty}{\infty}$" type.

$$\lim_{x \to \infty} \frac{\log x}{x} = \lim_{x \to \infty} \frac{(\log x)'}{x'} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0.$$

By introducing new variables, we may reduce to the case that: $g, f : (a, b) \to \mathbb{R}$. differentiable, we are interested in

$$\lim_{x \to a^+} \frac{f(x)}{g(x)}.$$

<u>Thm</u>: Assume $f, g : (a, b) \to \mathbb{R}$ differentiable, $g(x) \neq 0$ over (a, b). If either of the following condition is true

(1) $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$.

(2). $\lim_{x \to a} g(x) = +\infty$

And if

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A \in \mathbb{R} \cup \{+\infty, -\infty\}$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = A.$$

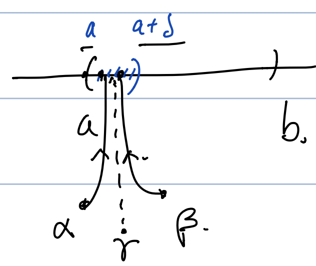
<u>Pf</u>: Here we only consider the case that $A \in \mathbb{R}$, the case $A = +\infty$ or $-\infty$ is an exercise.

$\because \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = A \quad \therefore$

For any $\varepsilon > 0$, $\exists\, \delta > 0$, s.t. $\forall x \in (a, a+\delta)$, we have.

$$\left| \frac{f'(x)}{g'(x)} - A \right| < \varepsilon.$$

$$\Leftrightarrow \quad A - \varepsilon < \frac{f'(x)}{g'(x)} < A + \varepsilon \qquad \forall x \in (a, a+\delta).$$



$\forall\, a < \alpha < \beta < a+\delta$, we have. $\gamma \in (\alpha, \beta)$

$$[f(\beta) - f(\alpha)] \cdot g'(\gamma) = [g(\beta) - g(\alpha)]\, f'(\gamma).$$

Then if $g(\beta) \neq g(\alpha)$.

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\gamma)}{g'(\gamma)} \in (A-\varepsilon, A+\varepsilon). \quad (*)$$

<u>case (1)</u>: $\lim_{\alpha \to a} g(\alpha) = 0$, $\lim_{\alpha \to a} f(\alpha) = 0$.

Take limit of $\alpha \to a$, we have.

$$\lim_{\alpha \to a} \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f(\beta)-0}{g(\beta)-0} = \frac{f(\beta)}{g(\beta)} \in [A-\varepsilon, A+\varepsilon].$$

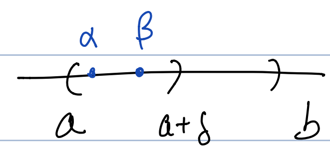
Thus, $\forall \varepsilon > 0$, we have found a $\delta > 0$, s.t. $\forall \beta \in (a, a+\delta)$, $\frac{f(\beta)}{g(\beta)} \in [A-\varepsilon, A+\varepsilon]$. This is precisely the definition that

$$\lim_{\beta \to a} \frac{f(\beta)}{g(\beta)} = A.$$

<u>case (2)</u> : $\lim_{\alpha \to a} g(\alpha) = +\infty$

We pick $\beta \in (a, a+\delta)$, then pick $\alpha \in (a, \beta)$ and α is sufficiently close to a, such that

$g(\alpha) > g(\beta)$.



$$\Rightarrow \quad \frac{g(\alpha)-g(\beta)}{g(\alpha)} > 0.$$

multiply $\frac{g(\alpha)-g(\beta)}{g(\alpha)}$ to $(*)$.

$$(A-\varepsilon)\frac{g(\alpha)-g(\beta)}{g(\alpha)} < \underbrace{\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} \cdot \frac{g(\alpha)-g(\beta)}{g(\alpha)}}_{\frac{f(\beta)-f(\alpha)}{g(\alpha)}} < (A+\varepsilon)\cdot\frac{g(\alpha)-g(\beta)}{g(\alpha)}$$

Take limit $\alpha \to a$, then. $\dfrac{g(\alpha) - g(\beta)}{g(\alpha)} \to 1.$, Hence.

$$A - \varepsilon \le \liminf_{\alpha \to a} \frac{f(\alpha) - f(\beta)}{g(\alpha)} \le \limsup_{\alpha \to a} \frac{f(\alpha) - f(\beta)}{g(\alpha)} \le A + \varepsilon$$

Take limit, $\varepsilon \to 0$, we get. $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = A$. #.