Differentiation / Derivation:

- definition. / example /.
- Leibniz rule / chain rules.
- mean value property.

• **Def**: Let $f: [a,b] \to \mathbb{R}$. We say f is differentiable at a point $p \in [a,b]$, if the following limit exists

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

← this is a function of x on $[a,b] \setminus \{p\}$

we denote this by $f'(p)$.

Let $f: [a,b] \to \mathbb{R}$.

Prop: If f is diff'ble at $p \in [a,b]$, then f is continuous at p.

Pf: We only need to show $\lim_{x \to p} f(x) = f(p)$.

$$\Leftrightarrow \lim_{x \to p} (f(x) - f(p)) = 0$$

$$\Leftrightarrow \lim_{x \to p} \left(\frac{f(x) - f(p)}{x - p} \right) \cdot (x - p) = 0$$

$$\Leftarrow \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = f'(p) \text{ exists.}$$

$$\because \lim_{x \to p} \left(\frac{f(x) - f(p)}{x - p} \cdot (x - p) \right) = \left(\lim_{x \to p} \frac{f(x) - f(p)}{x - p} \right) \cdot \left(\lim_{x \to p} (x - p) \right)$$

$$= f'(p) \cdot 0 = 0.$$

#.

Prop: If $f(x)$ is differentiable at p, then $\exists\, u(x)$, s.t.

$$f(x) = f(p) + (x-p)\cdot f'(p) + \underbrace{(x-p)\,u(x)}_{\text{remainder}}.$$

where $\lim_{x\to p} u(x) = 0$.

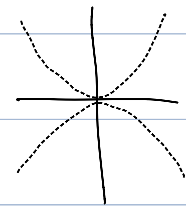
Pf: define

$$u(x) = \begin{cases} \dfrac{f(x)-f(p)}{x-p} - f'(p) & x \neq p \\ 0 & x = p. \end{cases}$$

#.

Ex :

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ -x^2. & x \in \mathbb{Q}^c. \end{cases}$$

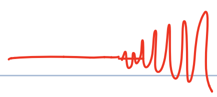


$f'(0)$ exist. , $f(x)$ is continuous at $x=0$.

$\forall x \neq 0$ $f'(x)$ doesn't exist, $f(x)$ is not cont.

Ex:

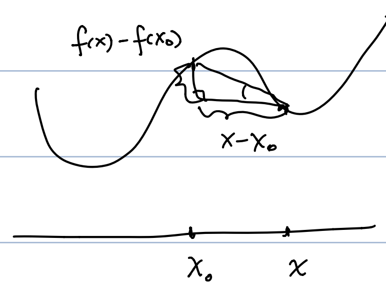
$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0. \end{cases}$$



- f is continuous everywhere.
- $f'(x)$ exists for any $x \neq 0$.

Use definition: consider the difference quotient (at $x=0$).

$$g(x) = \frac{f(x) - f(0)}{x - 0}.$$

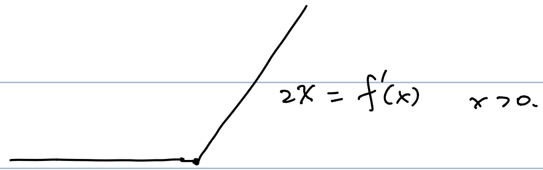


$$g(x) = \begin{cases} \dfrac{x\sin\left(\frac{1}{x}\right) - 0}{x - 0} = \sin\left(\frac{1}{x}\right). & x > 0 \\ \dfrac{0}{x} = 0 & x < 0 \end{cases}$$

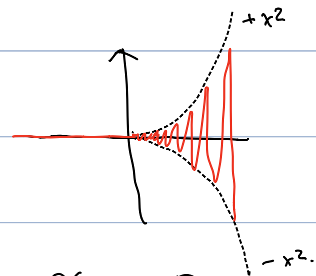
$\lim_{x\to 0^+} g(x)$ doesn't exist. $\Rightarrow$ $f'(0)$ does not exist.

• Ex:

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \le 0 \end{cases}$$

$f'(x) =$ 

• Q: if $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, and $f'(x)$ exists at all $x \in \mathbb{R}$. Is it automatic that f' is continuous?



Ans: No.

Ex:

$$f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right). & x > 0. \\ 0 & x \le 0 \end{cases}$$

$$f'(0+) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x \to 0^+} x \cdot \sin\left(\frac{1}{x}\right) = 0$$

$$f'(0-) = 0$$

$\Rightarrow$ $f'(0)$ exists and $f'(0) = 0$.

for $x > 0$,

$$\begin{aligned} f'(x) &= 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \left(\sin\left(\frac{1}{x}\right)\right)' \\ &= 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \left(-\frac{1}{x^2}\right) \cdot \cos\left(\frac{1}{x}\right). \\ &= 2x \cdot \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \end{aligned}$$

$$\lim_{x \to 0^+} f'(x) \neq 0.$$

Prop: Let $f, g : [a,b] \to \mathbb{R}$. And assume f, g are diff'ble at p. Then

① $(f+g)'(p) = f'(p) + g'(p)$

Leibniz. ② $(f \cdot g)'(p) = f'(p)\, g(p) + f(p) \cdot g'(p)$

③ If $g(p) \neq 0$, then. $(f/g)'(p) = \dfrac{f'g - f \cdot g'}{g^2}$.

Pf: ②: $\displaystyle\lim_{x \to p} \frac{f(x)\, g(x) - f(p) \cdot g(p)}{x - p} = \lim_{x \to p} \frac{f(x)\,(g(x) - g(p)) + f(x)\, g(p) - f(p)\, g(p)}{x - p}$

$$= \lim_{x \to p} f(x) \cdot \frac{g(x) - g(p)}{x - p} + \lim_{x \to p} \frac{(f(x) - f(p))}{x - p} \cdot g(p).$$

$$= f(p) \cdot g'(p) + f'(p) \cdot g(p).$$

\#. $A, B \subset \mathbb{R}$

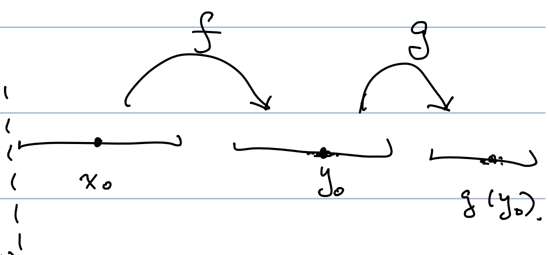
(more generally, $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$. s.t. $f(A) \subset B$.)

Chain Rule: Let $f, g : \mathbb{R} \to \mathbb{R}$.
assume $f(x_0) = y_0$, and f is diff'ble at x_0, g is diff'ble at y_0. Then the composition $h(x) = g(f(x))$ is diff'ble at x_0,

$$h'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

Pf: $\displaystyle\lim_{x \to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}.$

Since $f'(x_0)$ exist, we can write.



$\forall x \in \mathbb{R}$. $f(x) = f(x_0) + f'(x_0) \cdot (x - x_0) + (x - x_0) \cdot u(x)$

Similarly. $g(y) = g(y_0) + g'(y_0)(y - y_0) + (y - y_0)\, v(y)$

and $\lim_{x\to x_0} u(x)=0$, $\lim_{y\to y_0} v(y)=0$, $y_0=f(x_0)$.

Then. $g(f(x)) - g(y_0) = (f(x)-y_0)\cdot(g'(y_0)+v(f(x)))$

$= (f(x)-f(x_0))\,(g'(y_0)+v(f(x)))$

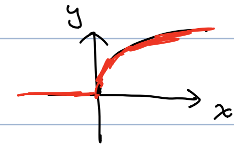
$= (x-x_0)\,(f'(x_0)+u(x))\cdot(g'(y_0)+v(f(x)))$

Hence $\lim_{x\to x_0}\dfrac{g(f(x))-g(f(x_0))}{x-x_0} = \lim_{x\to x_0}[f'(x_0)+u(x)][g'(f(x_0))+v(f(x))]$

$= f'(x_0)\cdot g'(f(x_0))$. #.

$y=\sqrt{x}$

$\Leftrightarrow$ $y^2=x$



Ex: $f(x)=\begin{cases}\sqrt{x} & x>0\\ 0. & x\le 0\end{cases}$

- is f continuous at $x=0$? $\lim_{x\to 0^+} f(x) = 0. = \lim_{x\to 0^-} f(x)$.

$f'(0)$ doesn't exist. , $f'(x)$ exist $\forall x\neq 0$.

$f'(x) = \begin{cases}\frac{1}{2}\frac{1}{\sqrt{x}} & x>0\\ 0 & x<0.\end{cases}$

$\forall p\in\mathbb{R}$:

- $(x^p)' = p\cdot x^{p-1}$. (for $x>0$).

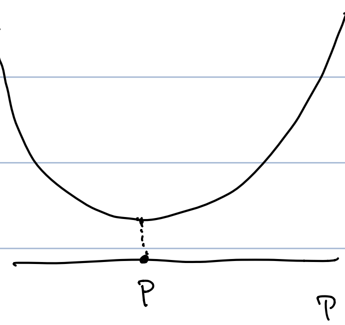
- $\forall n\in\mathbb{N}$, $(x^n)' = n\cdot x^{n-1}$.

⊕ Mean Value Theorem.

Def: Let $f: [a,b]\to\mathbb{R}$. (may not be cont.).

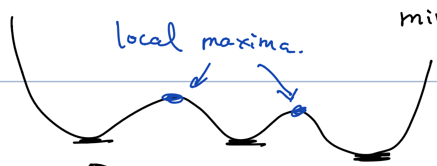
We say $p\in[a,b]$ is a local maximum of f if there is a $\delta>0$, s.t.

$f=$

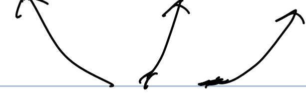


p is the (global) mininum of f.

f:



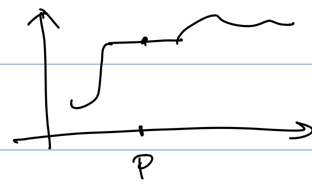
$\forall x \in [a,b] \cap B_\delta(p)$, we have $f(x) \le f(p)$.



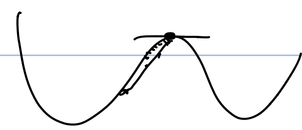
these are local minimums.

Similarly, we can define local min. ---

<u>Rmk</u>: it is possible that f is "locally constant" at p, then p is both a local max and local min.

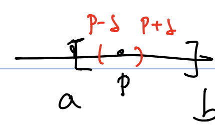


<u>Lemma</u>: Let $f: [a,b] \to \mathbb{R}$. If p is a local maximum of f, $p \in (a,b)$, and $f'(p)$ exists, then $f'(p) = 0$.



<u>Pf</u>: $f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$.

Since p is a local max of f, $\exists\, \delta > 0$, s.t. $\forall x \in [a,b] \cap B_\delta(p)$, $f(x) \le f(p)$. Since $p \in (a,b)$, we may take δ even smaller, s.t. $B_\delta(p) \subset [a,b]$. Then, if $x \in (p-\delta, p)$, then



$$\frac{f(x) - f(p)}{x - p} \ge 0 \qquad \because\ x - p < 0,\ f(x) - f(p) \le 0.$$

$$\Rightarrow \lim_{x \to p^-} \frac{f(x) - f(p)}{x - p} \ge 0.$$

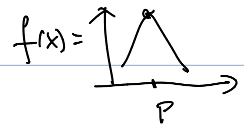
Similarly, $\lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} \le 0$.

Since $f'(p) = \lim_{x \to p} \frac{f(x) - f(p)}{x - p}$ exists, $\therefore f'(p) \geq 0$, $f'(p) \leq 0$

$\Rightarrow f'(p) = 0$. #.

Rmk: why we need the condition $p \in (a, b)$, and $f'(p)$ exist?

- p can be a local max, and $f'(p)$ non-existent.

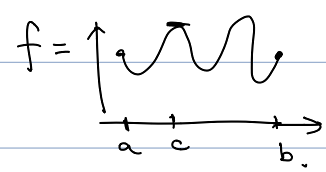
  e.g. 

-  local max are at the endpoints.

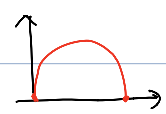
  the slope there are not zero.

  Hence to use this Lemma, only "interior" local max can be used.

Thm (Rolle) Let $f: [a, b] \to \mathbb{R}$ be a continuous function, assume $f'(x)$ exists for all $x \in (a, b)$. If $f(a) = f(b)$, Then, $\exists\ c \in (a, b)$, such that $f'(c) = 0$.



Rmk: ① $[a, b] \subset \mathbb{R}$ is compact. hence $f([a, b])$ is compact.

② $f(x) = \sqrt{x(1-x)}$  on $[0, 1]$ continuous.

$f'(x)$ does not exist for $x = 0, 1$.

Pf: ① if $f([a, b])$ is a single point, then f is a const function, any $c \in (a, b)$ has $f'(c) = 0$.

② If $\max(f([a, b])) \neq f(a)$, then Let $p \in (a, b)$.

s.t. $f(p) = \max(f([a,b]))$. Then by Lemma,

$f'(p) = 0$. Let $c = p$.

③ If $\min(f([a,b])) \neq f(a)$, then similar argument shows $f'(p) = 0$.

#.

Thm (Generalized Mean Value Theorem):

If $f, g : [a,b] \to \mathbb{R}$, are continuous, and differentiable on (a,b), then there exist $c \in (a,b)$, such that

$$(*) \quad [f(b) - f(a)] \cdot g'(c) = [g(b) - g(a)] \cdot f'(c).$$

Pf: Define

$$h(x) = [f(b) - f(a)][g(x) - g(a)] - [g(b) - g(a)][f(x) - f(a)]$$

Then $h(a) = 0$, $h(b) = 0$,

Hence, by Rolle' Thm $\exists\, c \in (a,b)$, s.t. $h'(c) = 0$.

$\Leftrightarrow (*)$ holds at c.

$$h'(x) = (f(b) - f(a))\, g'(x) - (g(b) - g(a))\, f'(x).$$ #.

Thm (Mean Value Thm): Let $f : [a,b] \to \mathbb{R}$. diff'ble on (a,b), then $\exists\, c \in (a,b)$, s.t.

$$f(b) - f(a) = (b-a) \cdot f'(c).$$

Pf: Use Generalized M.V.T., set $g(x) = x$.