Derivatives.

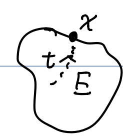
- Definition / Example.
- Leibniz rule. / Chain rule.
- Mean Value Theorem.  , 

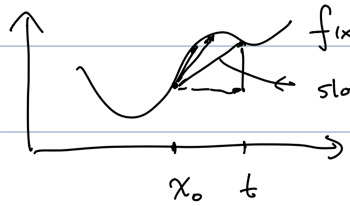
- Let $f : [a,b] \to \mathbb{R}$ be a real valued function. Define. $\forall x \in [a,b]$

$$f'(x) = \lim_{t \to x} \left(\frac{f(t) - f(x)}{t - x} \right)$$

This limit may not exist for all points. If $f'(x)$ exists, we say f is differentiable at x.

Recall: limit of a function at a point, $g : E \to \mathbb{R}$.

suppose x is a limit point of E, $\lim_{t \to x} g(t)$ 

$\forall x_0 \in [a,b]$,
$$g_{x_0}(t) = \frac{f(t) - f(x_0)}{t - x_0}$$


defined for $t \in [a,b] \setminus \{x_0\}$.

- Prop: if $f : [a,b] \to \mathbb{R}$ is differentiable at $x_0 \in [a,b]$, then f is continuous at x_0. i.e. $\lim_{x \to x_0} f(x) = f(x_0)$.

Pf:
$$f(x) - f(x_0) = \underbrace{\frac{f(x) - f(x_0)}{x - x_0}} \cdot \underbrace{(x - x_0)}.$$

Hence.
$$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim (x - x_0).$$

$$= f'(x_0) \cdot 0 = 0. \quad \#$$

Rmk: • If $f(x)$ is differentiable at x_0, it may happen that $f(x)$ is only continuous at x_0, not at any nearby points.

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ -x^2 & x \in \mathbb{Q}^c \quad \text{irrational.} \end{cases}$$

$f'(0) = 0.$, indeed $f(x)$ is cont. at $x=0$.
but $f(x)$ is not continuous at $x \in \mathbb{R} \setminus \{0\}$.

Ex: • $f(x) = x^2$. Compute $f'(3)$.

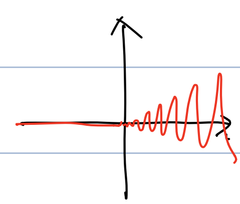
need to construct the "difference quotient"

$$g(x) = \frac{f(x) - f(3)}{x-3} = \frac{x^2 - 3^2}{x-3} = \frac{(x+3)(x-3)}{x-3} = x+3.$$

$$\Rightarrow \lim_{x \to 3} g(x) = 6. \qquad f'(3) = 6$$

• $$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Does $f'(0)$ exist?



for $x > 0$.

$$g(x) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \underline{\sin\left(\frac{1}{x}\right)}.$$

for $x < 0$, $g(x) = \frac{f(x)}{x} = 0.$

$\lim_{x \to 0^+} g(x)$ does not exist. Hence $f'(0)$ doesn't exist.

- $$f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & x > 0 \\ 0. & x \leq 0 \end{cases}$$

$f'(0)$ exists, $f'(0) = 0$.

<u>Rmk</u>: $f'(x)$ exits at all $x \in \mathbb{R}$, but $f'(x)$ is discontinuous at $x = 0$.

For $x > 0$, (to be proved later).

$$f'(x) = 2x \cdot \sin\left(\frac{1}{x}\right) + x^2 \cdot \left(-\frac{1}{x^2}\right) \cdot \cos\left(\frac{1}{x}\right).$$

$$= \underbrace{2x \cdot \sin\left(\frac{1}{x}\right)}_{\substack{\to 0 \\ \text{as } x \to 0^+}} - \underbrace{\cos\left(\frac{1}{x}\right)}_{\text{oscillates}}.$$

$f'(x)$ does not converge to $f'(0)$ as $x \to 0^+$.

<u>Thm</u>: Let $f, g : [a, b] \to \mathbb{R}$. Assume that f, g are differentiable at point $x_0 \in [a, b]$, then.

① $\forall c \in \mathbb{R}$, $(c \cdot f)'(x_0) = c \cdot f'(x_0)$

② $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

③ $(fg)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$.

④ If $g(x_0) \neq 0$, then $(f/g)'(x_0) = \dfrac{f'g - f \cdot g'|_{x_0}}{g^2(x_0)}$

<u>Pf</u>: ① $\displaystyle\lim_{x \to x_0} \frac{c \cdot f(x) - c \cdot f(x_0)}{x - x_0} = \lim_{x \to x_0} c \cdot \frac{f(x) - f(x_0)}{x - x_0} = c \cdot f'(x_0).$

③ $\displaystyle\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0}.$

$$f(x)\cdot g(x) - f(x_0)\,g(x_0) = [f(x)-f(x_0)+f(x_0)]\cdot[g(x)-g(x_0)+g(x_0)] - f(x_0)\,g(x_0)$$

$$= [f(x)-f(x_0)][g(x)-g(x_0)] + [f(x)-f(x_0)]\cdot g(x_0) + f(x_0)[g(x)-g(x_0)]$$

$$\lim_{x\to x_0} \frac{(f(x)-f(x_0))(g(x)-g(x_0))}{(x-x_0)} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}\cdot(g(x)-g(x_0))$$

$$= f'(x_0)\cdot 0 = 0.$$

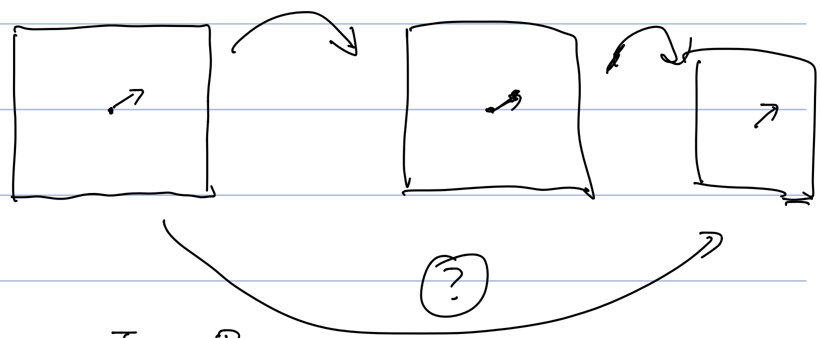
$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}\cdot g(x_0) = f'(x_0)\cdot g(x_0).$$

$$\lim_{x\to x_0.} f(x_0)\cdot\frac{g(x)-g(x_0)}{x-x_0} = f(x_0)\cdot g'(x_0).$$

②, ④ exercise.

<u>Rmk</u>: Leibniz rule : $(fg)' = f'\cdot g + f\cdot g'$.

<u>Chain Rule</u>:



<u>Thm</u>: Suppose $f: [a,b]\to\mathbb{R}$.

and $g: I \to \mathbb{R}$, $I\subset\mathbb{R}$.

Suppose for some $x_0\in[a,b]$. $f(x_0)=y_0$, $f([a,b])\subset I$.

Suppose $f'(x_0)$ and $g'(y_0)$ exists. Then, the composition

$h = g\circ f : [a,b]\to\mathbb{R}$. $h(x) := g(f(x))$.

is differentiable at x_0.

$$h'(x_0) = g'(y_0)\cdot f'(x_0).$$

idea: • form difference quotient:

$$\frac{h(x)-h(x_0)}{x-x_0} = \frac{g(f(x))-g(f(x_0))}{x-x_0} = \frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)} \cdot \frac{f(x)-f(x_0)}{x-x_0}$$

as $x \to x_0$, $f(x) \to f(x_0)$.

Trouble: $f(x)$ may equal to $f(x_0)$ near $x = x_0$.

• If $f(x)$ is differentiable at x_0, then.

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0).$$

$$\Rightarrow \lim_{x \to x_0} \overbrace{\frac{[f(x)-f(x_0)]}{x-x_0} - f'(x_0)}^{R_f(x).} = 0$$

or.

$$f(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \underline{R_f(x)\cdot(x-x_0)},$$

with $\lim_{x \to x_0} R_f(x) = 0.$

Pf: Since $f(x)$ is diff. at x_0, $g(x)$ is diff. at y_0.

we have.

$$(*) \quad f(x) - f(x_0) = (x-x_0)\cdot(f'(x_0) + \underline{u(x)})$$

$$(**) \quad g(y) - g(y_0) = (y-y_0)\,(g'(y_0) + \underline{v(y)})$$

$$\lim_{x \to x_0} u(x) = 0, \qquad \lim_{y \to y_0} v(y) = 0.$$

$$h(x) - h(x_0) = g(\underline{f(x)}) - g(\underline{f(x_0)}).$$

$$\overset{(**)}{=} \left(f(x) - f(x_0)\right)\left(g'(f(x_0)) + \underline{v(f(x))}\right)$$

$$\overset{(*)}{=} (x - x_0)\cdot\left(f'(x_0) + u(x)\right)\cdot\left(g'(f(x_0)) + v(f(x))\right).$$

Thus,

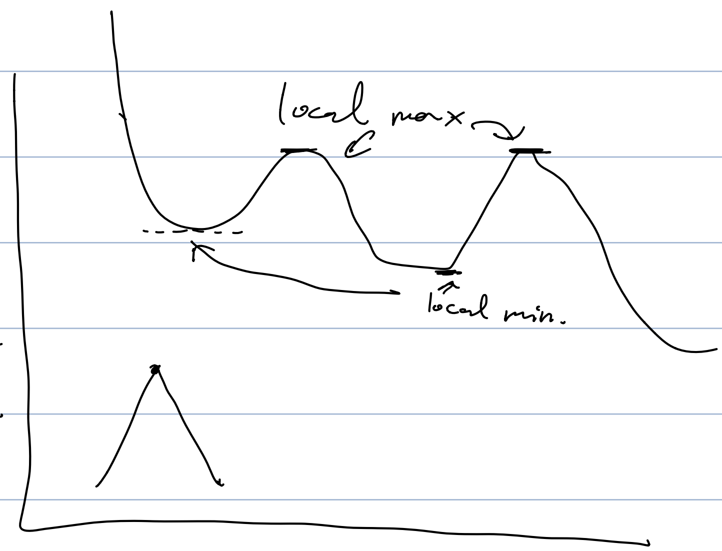
$$\lim_{x\to x_0} \frac{h(x) - h(x_0)}{x - x_0} = \lim_{x\to x_0} \left(f'(x_0) + u(\cdots)\right)\left(g'(f(x_0)) + v(\cdots)\right)$$

$$= f'(x_0)\cdot g'(f(x_0)) \qquad \#.$$

Ex: $h(x) = \sin(x^2)$ $\qquad x \xmapsto{f} x^2 \xmapsto{g} \sin(x^2)$.

$f(x) = x^2$, $\qquad g(y) = \sin(y)$.

$$h'(x) = f'(x)\cdot g'(f(x)).$$
$$= (2x)\cdot\cos(x^2).$$



Mean Value Theorem:

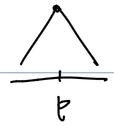
Def: Say $f: [a,b] \to \mathbb{R}$.

We say f has a local maximum at point $p \in [a,b]$, if $\exists\, \delta > 0$, and $\forall x \in [a,b] \cap B_\delta(p)$,

$$f(x) \le f(p).$$

Prop: Let $f: [a,b] \to \mathbb{R}$. If f has local max at $p \in (a,b)$, and if $f'(p)$ exists, then $f'(p) = 0$.

Rmk: Ex: $f(x) = x^2$, $\qquad x \in [-1,1]$. $\quad$ local max is at $x = \pm 1$, the endpoints, here $f'(x) \neq 0$.

Ex : $f(x) =$  $f'(p)$ doesn't exist.

Pf : • Since p is a local max for f, $\exists\, \delta > 0$, s.t. $(p-\delta, p+\delta) \subset [a,b]$.

and $f(p)$ is the max of $f|_{(p-\delta,\, p+\delta)}$.

• Let. $g(x) = \dfrac{f(x) - f(p)}{x - p}$ for $x \in [a,b] \setminus \{p\}$.

If $x \in (p-\delta, p)$, then. $g(x) \geq 0$. $\Leftarrow \begin{cases} x-p < 0 \\ f(x) - f(p) \leq 0. \end{cases}$

hence $\lim\limits_{x \to p^-} g(x) \geq 0$.

Similarly. if $x \in (p, p+\delta)$, then $g(x) \leq 0$. $\Leftarrow \begin{cases} x-p > 0 \\ f(x)-f(p) \leq 0 \end{cases}$

$\lim\limits_{x \to p^+} g(x) \leq 0$.

Since. $\lim\limits_{x \to p} g(x)$ exist. $\Rightarrow$ $\lim\limits_{x \to p} g(x) = \lim\limits_{x \to p^+} g(x) = \lim\limits_{x \to p^-} g(x)$

hence $\lim\limits_{x \to p} g(x) = 0$.

(Rolle)

Thm : Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function.

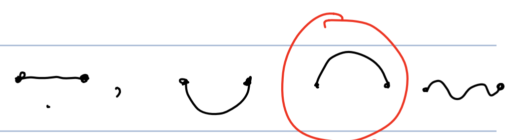
and. f is differentiable in (a,b). If $f(a) = f(b)$,

then there is some $c \in (a,b)$, such that $f'(c) = 0$.

Pf : • if $f([a,b])$ is a single point, then f

is a constant function, then one can take

c to be any pt in (a,b).



if f is not a constant function,

• f has a maximum or minimum. whose value is

this case.

different with the end point $f(a) = f(b)$. WLOG, Suppose $p \in (a,b)$.

and $f(p) = \max(f([a,b]))$, then. p is also

a local max, hence by previous prop, $f'(p) = 0$. #.