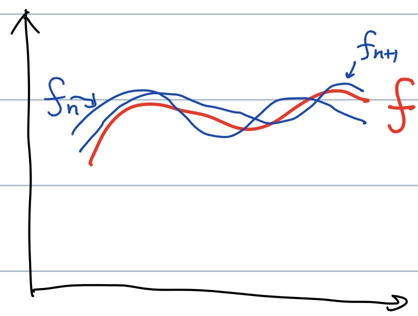
1. Let $f, f_n : [a,b] \to \mathbb{R}$ be a seq of real valued functions.

We say $f_n \to f$ ~~conv~~ <u>uniformly convergent</u>, if
$$\lim_{n\to\infty} \left(\sup_{x\in[a,b]} |f_n(x) - f(x)| \right) = 0.$$

- If f_n continuous, $f_n \to f$ unif, then f is continuous.

<u>Question</u>: If f_n are integrable (or differentiable), is f also integrable (or differentiable, resp)?

<u>Thm 7.16</u> (Rudin). Let α be a monotone increasing fcn on $[a,b]$.
Let f_n be a seq of fcn on $[a,b]$, $f \in R(\alpha)$.
Assume $f_n \to f$ uniformly. Then $f \in R(\alpha)$, and
$$\int_a^b f \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha.$$



<u>Pf</u>: Let $\varepsilon > 0$ be fixed. Then $\exists N$, s.t. $\forall n > N$.
$$|f_n(x) - f(x)| < \varepsilon \qquad \forall n > N.$$
i.e. $f_n(x) - \varepsilon < f(x) < \varepsilon + f_n(x) \qquad \forall x \in [a,b]. \ \forall n > N.$

Consider upper and lower integrals, we have.
$$(*) \quad \underline{\int} (f_n - \varepsilon)\, d\alpha \le \underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} (f_n + \varepsilon)\, d\alpha$$

At this moment, we don't know if f is integrable or not, but f is bounded. hence $\overline{\int_a^b} f\, d\alpha$, $\underline{\int_a^b} f\, d\alpha$ exist.

Since f_n are integrable, hence,
$$\overline{\int} (f_n + \varepsilon)\, d\alpha - \underline{\int} (f_n - \varepsilon)\, d\alpha$$
$$= \int (f_n + \varepsilon)\, d\alpha - \int (f_n - \varepsilon)\, d\alpha = 2\varepsilon \cdot \int d\alpha = 2\varepsilon[\alpha(b) - \alpha(a)].$$
Hence $0 \le \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha \le 2\varepsilon \cdot [\alpha(b) - \alpha(a)]$.

Since $\varepsilon > 0$ is arbitrary, hence $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$ #.

Cor: If $\{f_n\}$ is a seq of integrable fcn w.r.t. α.

and if $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly, then.

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx.$$

Pf: Consider the partial sum $F_N(x) = \sum_{n=1}^{N} f_n(x)$,

and apply the theorem.

Unif Conv. and Differentiation:

- Ex: • $\{f_n\}$, differentiable, $f_n \to f$ uniformly, but f is not differentiable.

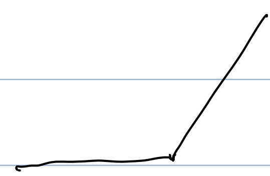
$$f_n(x) = \sqrt{x^2 + \frac{1}{n}} \approx |x|$$

f_n: differentiable ✓. $f_n \to |x|$ uniformly.

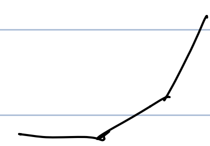
$$\sqrt{x^2+\frac{1}{n}} - \sqrt{x^2} = \frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}} + \sqrt{x^2}} \leq \frac{1}{n} \Big/ \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}}.$$

$\to 0.$

• $f_1(x) = \log(1+e^x).$



$f_n(x) = \frac{1}{n} \log(1+e^{nx})$



$f_n(x) \to \max\{0, x\}.$

$\frac{1}{n}\log 1 \qquad \frac{1}{n}\log(e^{nx})$

• $f_n(x) = \frac{1}{n}\cdot \sin(n^2 \cdot x).$ $\qquad f_n'(x) = n \cdot \cos(n^2 x)$

f_n converges uniformly, $\qquad f_n'(x)$ doesn't converge.

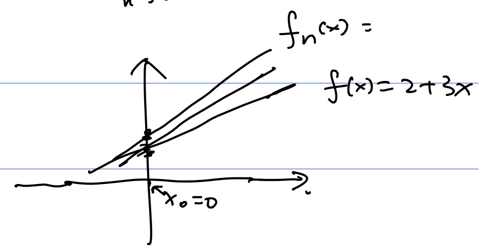
Thm: Let f_n be a sequence of differentiable fcn on. $[a,b]$, assume that ① $\exists x_0 \in [a,b]$, s.t. $f_n(x_0) \to C$

② $\{f_n'(x)\}$ converges uniformly on $[a,b]$.

Then., $f_n \to f$ uniformly. and $f'(x) = \lim_{n\to\infty} f_n'(x)$ $\forall x \in [a,b]$

Ex: $f_n(x) = (2+\frac{1}{n}) + (3+\frac{1}{n}) \cdot x$

over $[-1,1]$. , we have $f_n \to f$ uniformly.



idea: use integration.

$x > x_0$, Naively:

$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt.$

but the problem is that, we don't know if f' is integrable or not.

Pf: Fix $\varepsilon > 0$. $\exists N > 0$, s.t.

① $\forall n,m > N$,

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$$

② $\forall n,m > N$,

$$\sup_{x\in[a,b]} |f_n'(x) - f_m'(x)| < \frac{\varepsilon}{2(b-a)}$$

Fix $n,m > N$. Let $f_{n,m}(x) := f_n(x) - f_m(x)$.

Apply mean value theorem to $f_{n,m}(x)$, on an interval $[x,t]$ (if $x<t$) or ($[t,x]$, if $t<x$) in $[a,b]$.

$$f_{n,m}(x) - f_{n,m}(t) = f_{n,m}'(\xi)\,(x-t)$$

for $\xi \in (x,t)$. Since $|f_{n,m}'(\xi)| < \frac{\varepsilon}{2(b-a)}$. we have.

$(*)$. $\quad |f_{n,m}(x) - f_{n,m}(t)| \leq \frac{\varepsilon}{2(b-a)} \cdot |x-t|$

Step 1: Prove $\{f_n(x)\}$ $\underbrace{\text{Cauchy}}^{\text{uniformly}}$,

$\forall x \in [a,b]$

$$|f_n(x) - f_m(x)| = |f_{n,m}(x)| \leq |f_{n,m}(x) - f_{n,m}(x_0)| + |f_{n,m}(x_0)|$$

$$\leq \frac{\varepsilon}{2(b-a)} \cdot |x-x_0| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

since this is true $\forall n,m > N$, Hence $\{f_n\}$ is uniformly Cauchy, hence $\exists f$, that $f_n \to f$ uniformly on $[a,b]$.

Step 2: Prove $f'(x)$ exists. and equals $\lim_{n\to\infty} f_n'(x)$.

- Fix an $x \in [a,b]$, define.

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}, \qquad \phi(t) = \frac{f(t) - f(x)}{t-x}$$

well defined functions for $t \in [a,b] \setminus \{x\}$.

- since $f_n(t) \to f(t)$, $f_n(x) \to f(x)$ as $n \to \infty$, hence $\phi_n(t) \to \phi(t)$ pointwise $\forall t \in [a,b] \setminus \{x\}$.

  we want to prove unif conv.

- Consider. $|\phi_n(t) - \phi_m(t)| = \frac{1}{|t-x|} |f_n(t) - f_n(x) - (f_m(t) - f_m(x))|$

$$= \frac{1}{|t-x|} |f_n(t) - f_m(t) - (f_n(x) - f_m(x))|$$

$$\leq \frac{1}{|t-x|} \cdot \frac{\varepsilon}{2(b-a)} \cdot |t-x| = \frac{\varepsilon}{2(b-a)} \qquad \forall n,m > N.$$

Hence. $\{\phi_n\}$ is uniformly Cauchy for $t \in [a,b] \setminus \{x\}$. $\Rightarrow$ $\phi_n \to \phi$ uniformly.

Hence

$$\underline{\lim_{t\to x} \phi(t)} = \lim_{t\to x} \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{n\to\infty} f_n'(x).$$

the existence of $\lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ shows $f'(x)$ exist.

Hence

$$f'(x) = \lim_{t\to x} \phi(t) = \lim_{n\to\infty} f_n'(x). \qquad \#.$$