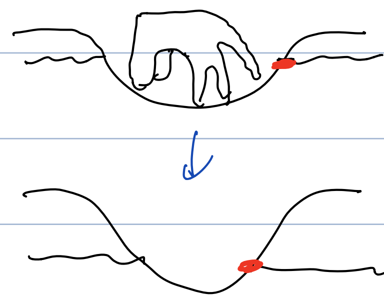
Today: Finish Ch 6.

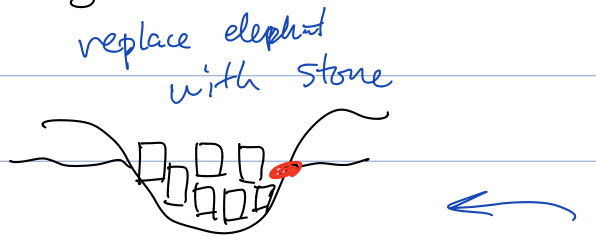
Next time: finish Ch 7. (unif convergence + $\frac{d}{dx}$, $\int$.).

Review problems.

- General Comments of integral:
  - Archaemede --- Eureka : "volume of crown"
  - 曹冲 Three Kingdom time (Han dynasty)

    weight an elephant.





Rudin. Thm 6.17 ~ Thm 6.21.

- Stieltjes integral assigns weight to an interval $I = [c, d]$, $\alpha(I) = \alpha(d) - \alpha(c)$. Integral $\int_a^b f \, d\alpha$ is approximated by "weighted" sum.

$$\sum_i \underbrace{\text{"}f(I_i)\text{"}} \cdot \alpha(I_i)$$

can be $\sup_{I_i} f(x)$, $\inf_{I_i} f(x)$, or $f(s_i)$ $s_i \in I_i$ sampling.

In special cases, the weight are given by a "density function", $\alpha(I) = \int_c^d \alpha'(x)\,dx$, in this case, we want to say

$$\underbrace{\int f \, d\alpha}_{\text{stieltjes integral}} = \underbrace{\int f \cdot \alpha' \, dx}_{\text{Rieman integral}}$$

Thm 6.17: Let α be a monotone increasing function on $[a,b]$ such that α' exists and is Riemann integrable $\left(\text{w.r.t } dx\right)$. Then for any bounded real function f on $[a,b]$,

$$f \in R(\alpha) \iff f \cdot \alpha' \in R$$

In this case,

$$\int_a^b f \, d\alpha = \int_a^b f \cdot \alpha' \cdot dx.$$

Pf: idea we will show:

$$\overline{\int_a^b} f \, d\alpha = \overline{\int_a^b} f \cdot \alpha' \, dx \qquad (*)$$

and

$$\underline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \cdot \alpha' \cdot dx.$$

then this implies the conclusion. I will only prove $(*)$.

Fix an $\varepsilon > 0$. Since $\alpha' \in R$, then $\exists$ partition P

$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, such that

$$(**) \qquad U(P, \alpha') - L(P, \alpha') < \varepsilon \qquad I_i = [x_{i-1}, x_i]$$

recall.

- For any sample points $\{s_i \in I_i\}$, under $(**)$,

$$L(P, \alpha') \le \sum_i \alpha'(s_i) \cdot \Delta x_i \le U(P, \alpha')$$

- also recall, using mean value theorem for α on each I_i,

we get $\alpha(x_i) - \alpha(x_{i-1}) = (x_i - x_{i-1}) \cdot \alpha'(t_i)$, for some $t_i \in I_i$.

i.e. $\Delta \alpha_i = \alpha'(t_i) \cdot \Delta x_i$

- Now, we want to compare.

$\cdot\ \mathcal{U}(P, f, \alpha) = \sum_i \left(\sup_{I_i} f(x)\right) \cdot \Delta\alpha_i$

$\cdot\ \mathcal{U}(P, f\alpha') = \sum_i \left[\sup_{I_i} (f(x)\cdot\alpha'(x))\right] \cdot \Delta x_i$

can not compare the 2 sup directly. We use sample points to approximate.

- $\forall \{s_i \in I_i\}$ chosen sample points,

$$
\begin{aligned}
& \left| \sum_i f(s_i)\cdot\Delta\alpha_i - \sum_i f(s_i)\,\alpha'(s_i)\cdot\Delta x_i \right| \\
= & \left| \sum_i f(s_i)\cdot\alpha_i'(t_i)\cdot\Delta x_i - \sum_i f(s_i)\cdot\alpha'(s_i)\cdot\Delta x_i \right|. \\
= & \left| \sum_i f(s_i)\cdot(\alpha_i'(t_i) - \alpha_i'(s_i))\cdot\Delta x_i \right|. \\
\leq & \sum_i \cdot |f(s_i)| \cdot |\alpha_i'(t_i) - \alpha_i'(s_i)| \cdot \Delta x_i \\
\leq & \; M \cdot \underbrace{\sum_i |\alpha_i'(t_i) - \alpha_i'(s_i)|\cdot\Delta x_i}_{\leq \varepsilon.} \leq M\cdot\varepsilon.
\end{aligned}
$$

$M = \sup_{[a,b]} |f(x)|$

Hence. $\forall \{s_i \in I_i\}$ sample points.

(A). $\left| \sum_i f(s_i)\cdot\Delta\alpha_i - \sum f(s_i)\cdot\alpha'(s_i)\cdot\Delta x_i \right| \leq M\cdot\varepsilon.$

A $\Rightarrow$.

$$
\begin{aligned}
\sum_i f(s_i)\,\Delta\alpha_i &\leq M\varepsilon + \sum_i f(s_i)\cdot\alpha'(s_i)\cdot\Delta x_i \\
&\leq M\varepsilon\, + \mathcal{U}(P, f\alpha')
\end{aligned}
$$

Since this is true for $\{s_i \in I_i\}$. hence.

$\mathcal{U}(P, f, \alpha) = \sum_i \sup_{s_i \in I_i} f(s_i) \cdot \Delta\alpha_i \leq M\varepsilon + \mathcal{U}(P, f\alpha')$

Hence we get

(A1). $\mathcal{U}(P, f, \alpha) \leq M\varepsilon + \mathcal{U}(P, f\alpha')$

similarly,

$\sum f(s_i)\,\alpha'(s_i)\,\Delta x_i \leq M\varepsilon + \sum f(s_i)\cdot\Delta\alpha_i$

$$\leq M\varepsilon + \mathcal{U}(P, f, \alpha)$$

$\Rightarrow$ (A2). $\quad \mathcal{U}(P, f\alpha') \leq M\varepsilon + \mathcal{U}(P, f, \alpha).$

(A1)+(A2) $\Rightarrow \quad |\mathcal{U}(P, f, \alpha) - \mathcal{U}(P, f\alpha')| \leq M\varepsilon,$

To get rid of the partition P dependence, we use

definition of upper integral:

$$\overline{\int_a^b} f\, d\alpha = \inf_P \mathcal{U}(P, f, \alpha).$$

$\exists$ seq of partitions P_i, $P_i \supset P$.

$$= \lim_{i\to\infty} \mathcal{U}(P_i, f, \alpha).$$

by replacing P_i with $Q_i = P_i \cup P_{i-1} \cup \cdots \cup P_1$

we have sequence $Q_1, Q_2, \cdots$, s.t. $Q_n \supset P_n$. $Q_n \supset Q_{n-1}$.

$$= \lim_{n\to\infty} \mathcal{U}(Q_n, f, \alpha)$$

similarly, $\quad \overline{\int_a^b} f\alpha' \cdot dx = \lim_{n\to\infty} \mathcal{U}(\tilde{Q}_n, f\alpha')$

$\tilde{Q}_n$ partition of $[a,b]$, $\quad \tilde{Q}_n \supset \tilde{Q}_{n-1}$.

now, take common refinements. $S_n = Q_n \cup \tilde{Q}_n$. Thus.

$$\left|\overline{\int_a^b} f\, d\alpha - \overline{\int_a^b} f\alpha'\, dx\right| = \left|\lim_{n\to\infty} \mathcal{U}(S_n, f, \alpha) - \lim_{n\to\infty} \mathcal{U}(S_n, f\alpha')\right|$$

$$= \lim_{n\to\infty} |\mathcal{U}(S_n, f, \alpha) - \mathcal{U}(S_n, f\alpha')| \leq M\varepsilon.$$

Since LHS is indep of ε, and $\varepsilon > 0$ is arbitrary, hence,

LHS $= 0$, i.e. $\quad \overline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\alpha'\, dx.$ #

Ex: say on $[0,1]$, let $\alpha = x^2$. then $\alpha' = 2x$.

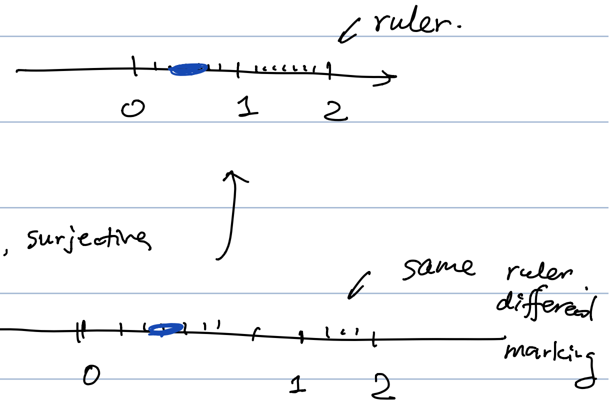
Thm $\Rightarrow$. $f \in \mathcal{R}(\alpha)$ iff. $f(x)\cdot(2x)$ is integrable.

"get rid of marking on the axis"

Thm. 6.19 (Change of variables).

Let α be increasing on $[a,b]$.

$f \in R(\alpha)$.



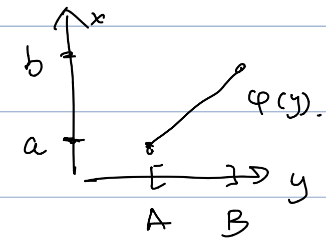
Assume we have a strictly ~~m~~ increasing, surjective function $\varphi : [A,B] \to [a,b]$.

and. $\beta : [A,B] \to \mathbb{R}$ given by $\beta = \alpha \circ \varphi$

$g : [A,B] \to \mathbb{R}$ —— $g = f \circ \varphi$.

then. $g \in R(\beta)$, and

$$\int_A^B g \, d\beta = \int_a^b f \, d\alpha.$$



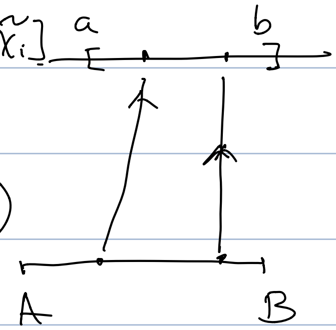
Pf: • φ is a bijection.

$\Rightarrow \{\text{partitions on } [a,b]\} \overset{1:1}{\longleftrightarrow} \{\text{partition on } [A,B]\}$.

$$P \longleftrightarrow \tilde{P}, \quad \tilde{x}_0$$

$$\{x_0 < x_1 < \cdots < x_n\} \longleftrightarrow \{\varphi^{-1}(x_0) < \varphi^{-1}(x_1) < \cdots < \varphi^{-1}(x_n)\}$$

$$I_i = [x_{i-1}, x_i] \longleftrightarrow \tilde{I}_i = [\tilde{x}_{i-1}, \tilde{x}_i]$$



• $$U(P,f,\alpha) = \sum \Big(\sup_{x \in I_i} f(x)\Big)\big(\alpha(x_i) - \alpha(x_{i-1})\big)$$

$$= \sum \Big(\sup_{y \in \tilde{I}_i} g(y)\Big) \cdot \big(\beta(\tilde{x}_i) - \beta(\tilde{x}_{i-1})\big),$$

$$= U(\tilde{P}, g, \beta).$$

similarly $L(P,f,\alpha) = L(\tilde{P}, g, \beta)$

$\Rightarrow$ g is integrable w.r.t. β. #.

Integration and Differentiation:

Thm: Let f be a Riemann integrable function on $[a,b]$, define

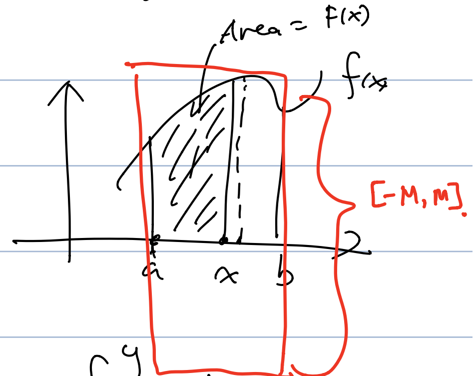
$$F(x) = \int_a^x f(u)\,du.$$

Then
- $F(x)$ is continuous.
- if $f(x)$ is continuous at x_0, then $F(x)$ is differentiable at x_0, $F'(x_0) = f(x_0)$.

If f is a smooth function on $[a,b]$, and

$$F(x) = \int_a^x f(u)\,du.$$

then F is smooth, with $F'(x) = f(x)$.



Pf: (1) for any $a \le x < y \le b$, we have

$$F(y) - F(x) = \int_x^y f(u)\,du$$

$$|F(y) - F(x)| \le \int_x^y |f(u)| \cdot du \le M \cdot \int_x^y du = M(y-x).$$

$M := \sup_{[a,b]} f(u)$

Hence F is Lipschitz continuous, hence continuous.

(2). Fix $\varepsilon > 0$. Since f is continuous at x_0, $\therefore \exists \delta > 0$. s.t. $\forall\, |x - x_0| < \delta$, $x \in [a,b]$, we have $|f(x) - f(x_0)| < \varepsilon$.

We want to show:

$$\lim_{x \to x_0^+} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0), \quad \text{similarly } x \to x_0^- \cdots \text{ the left limit}$$

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^x f(u)\,du.$$

$$f(x_0) = \frac{1}{x - x_0} \int_{x_0}^x f(x_0)\,du \quad \text{constant integrand.}$$

Hence
$$\left| \frac{F(x)-F(x_0)}{x-x_0} - f(x_0) \right| = \left| \frac{1}{x-x_0} \int_{x_0}^{x} \big(f(u)-f(x_0)\big)\cdot du \right|.$$

$$\leq \frac{1}{x-x_0} \int_{x_0}^{x} |f(u)-f(x_0)| \cdot du.$$

If $x_0 < x < x_0+\delta$, then $x_0 < u < x < x_0+\delta$, then $|f(u)-f(x_0)| < \varepsilon$.

$$\leq \frac{1}{x-x_0} \cdot \varepsilon \cdot \int_{x_0}^{x} du = \varepsilon.$$

Since for every ε, we can find such a $\delta > 0$. we have
$$\lim_{x \to x_0^+} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0).$$

Similarly for the left limit. #.

Thm (Fundamental Thm of Calculus):

Let F be a differentiable function on $[a,b]$, and assume $f = F'(x)$ is integrable, then.
$$\int_a^b f(x)\,dx = F(b)-F(a).$$

Rmk: it is possible to have a function F, s.t. $F'(x)$ exists $\forall x \in [a,b]$. and F' is bounded, but $F'(x)$ is not integrable. (wiki Volterra function)

Pf: • For any partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$,
$$F(b) - F(a) = F(x_n) - F(x_0)$$

$$= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + \cdots + F(x_1) - F(x_0)$$

$$= \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

- Since F is differentiable, $\exists t_i \in [x_{i-1}, x_i]$, s.t.

$$F(x_i) - F(x_{i-1}) = F'(t_i) \cdot (x_i - x_{i-1}). \quad \text{(M.V.T)}$$

- Since f is integrable, $\forall \varepsilon > 0$, $\exists P$, s.t.

$$U(P,f) - L(P,f) < \varepsilon.$$

Hence. $F(b) - F(a) = \sum_{i=1}^{n} F'(t_i) \cdot \Delta x_i$

$$= \sum_{i=1}^{n} f(t_i) \cdot \Delta x_i \qquad \text{for some } t_i \in [x_{i-1}, x_i]$$

$$\in [L(P,f), U(P,f)]$$

and since

$$\int_a^b f\,dx \in [L(P,f), U(P,f)].$$

$$\therefore \left| \int_a^b f\,dx - (F(b) - F(a)) \right| \le U(P,f) - L(P,f) < \varepsilon.$$

Since $\varepsilon > 0$ arbitrary, we get

$$\int_a^b f\,dx = F(b) - F(a). \qquad \#.$$

Thm (integration by part): On $[a,b]$, assume F, G are differentiable fcn. with $F' = f$, $G' = g$ integrable, then.

$$\int_a^b F\, dG = F\cdot G\Big|_a^b - \int_a^b G\cdot dF.$$

i.e.

$$\int_a^b F(x)\, g(x)\cdot dx = F(b)\,G(b) - F(a)\,G(a) - \int_a^b G(x)\, f(x)\, dx$$

Pf: Let $H = F\cdot G$, then differentiable $\because$ F, G are differentiable,

$$H' = F'\cdot G + F\cdot G',$$

$F' \in R$, G cont. $\Rightarrow$ G integrable $\Rightarrow$ $F'G$ integrable. similarly FG' integrable. $\Big\}$ $\therefore H'$ integ

Hence

$$\int_a^b H'\, dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b F'\cdot G + F\cdot G' \cdot dx = F(b)\,G(b) - F(a)\,G(a). \qquad \#$$