Today : • Finish the chapter on integration.

1. One motivation to consider Stieltjes integral, is that it allows for "general weight", namely each small interval $[c,d]$ is given a weight, $\alpha(d)-\alpha(c)$, or $\alpha([c,d])$, rather just their length. One special case to assign weight, is to use a "density function", $g(x)$, and $\alpha(d)-\alpha(c) = \int_c^d g(x)\,dx$, in other words, $g(x) = \alpha'(x)$.

Riemann integrable w.r.t. dx

Thm 6.17 : Assume α is increasing, and $\alpha' \in \mathcal{R}$ on $[a,b]$. And f a bounded real function on $[a,b]$. Then.

- $f \in \mathcal{R}(\alpha) \iff f\cdot\alpha' \in \mathcal{R}$.
- and in this case, $\int_a^b f\,d\alpha = \int_a^b f\cdot\alpha'\,dx$.

idea : • we are going to prove. $\mathcal{U}(f,\alpha) = \mathcal{U}(f\alpha')$

i.e. $\overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\alpha'\,dx$, and similarly

$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\alpha'\,dx$.

$\Rightarrow \quad \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$ iff $\overline{\int_a^b} f\alpha'\,dx = \underline{\int_a^b} f\alpha'\,dx$.

ex : $\alpha = x^2$, $f = x$.

then $\alpha' = 2x$.

$$\int_a^b x\cdot d(x^2) \overset{\text{Thm.}}{=\!=} \int_a^b \underbrace{x}_{f}\cdot\underbrace{2x}_{\alpha'}\cdot dx.$$

$\forall g$ bound real $\int_a^b g(x)\cdot dx^2$ exists iff $\int_a^b g(x)\cdot(2x)\cdot dx$ exists.

Pf: Let $\varepsilon > 0$ be given. Since $\alpha' \in \mathcal{R}$, $\exists\ P = \{x_0 < x_1 < \cdots < x_n\}$ partition, such that

$$(*) \qquad \mathcal{U}(P, \alpha') - L(P, \alpha') < \varepsilon.$$

[recall MVT: $f(b) - f(a) = (b-a)\cdot f'(\xi)$]

- By mean value theorem:

$$\Delta\alpha_i = \alpha'(t_i)\cdot\Delta x_i \quad \text{for some } t_i \in (x_{i-1}, x_i).$$

$I_i = [x_{i-1}, x_i]$.

- From the "sampling pointing thm": given any $\{s_i \in I_i\}$ and $\{t_i \in I_i\}$ sample points, and given $(*)$, we have.

$$\Sigma_i\, |\alpha'(t_i) - \alpha'(s_i)|\cdot\Delta x_i < \varepsilon.$$

- Since f is real and bounded, we have $M = \sup_{x\in[a,b]} |f(x)|$.

- Claim: For any set of sample points $\{s_i \in I_i,\ i = 1, \cdots, n\}$, we have.

$$(**) \qquad \left|\Sigma_i\, f(s_i)\cdot\Delta\alpha_i - \sum_i f(s_i)\cdot\alpha'(s_i)\cdot\Delta x_i\right| \leq M\varepsilon.$$

Pf of claim:

- $\Sigma_i\, f(s_i)\,[\Delta\alpha_i - \alpha'(s_i)\cdot\Delta x_i]$

$= \Sigma_i\, f(s_i)\cdot[\alpha'(t_i) - \alpha'(s_i)]\cdot\Delta x_i$

$\Rightarrow |\Sigma f(s_i)\Delta\alpha_i - \Sigma f(s_i)\,\alpha'(s_i)\cdot\Delta x_i| \leq \Sigma_i\, |f(s_i)|\cdot|\alpha'(t_i) - \alpha'(s_i)|\cdot\Delta x_i$

$\leq M\cdot\Sigma_i\, |\alpha'(t_i) - \alpha'(s_i)|\cdot\Delta x_i \leq M\cdot\varepsilon.$

upper bound for $|f|$

discrepancy between approximations of $d\alpha$ and $\alpha' dx$.

Hence the claim.

Given this claim, we have $\forall \{s_i \in I_i\}$,

$$(**) \Rightarrow \quad \sum_i f(s_i)\cdot \Delta\alpha_i \leq \underline{\sum_{i=1} f(s_i)\,\alpha'(s_i)\cdot \Delta x_i} + M\varepsilon.$$

put an upper bound on RHS.

$$\Rightarrow \quad \sum_i f(s_i)\,\Delta\alpha_i \leq U(P, f\alpha') + M\varepsilon.$$

sup over s_i for LHS.

$$\Rightarrow \quad U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon.$$

$$(**) \Rightarrow \quad \sum_i f(s_i)\cdot\alpha'(s_i)\cdot\Delta x_i \leq \sum f(s_i)\cdot\Delta\alpha_i + M\varepsilon.$$

$$\Rightarrow \quad U(P, f\alpha') \leq U(P, f, \alpha) + M\varepsilon.$$

Hence, the two inequality above implies.

$$(***) \quad |U(P, f\alpha') - U(P, f, \alpha)| \leq M\varepsilon.$$

- For any Q refinement of P, if we replace P by Q then $(*)$ holds and $(***)$ holds. Hence we can take a sequence of partitions $P_1, P_2, P_3, \ldots$ such P_{n+1} refines P_n, and $\lim_{i\to\infty} U(P_i, f, \alpha) = \overline{\int_a^b} f\,d\alpha$

and $\lim_{i\to\infty} U(P_i, f\alpha') = \overline{\int_a^b} f\alpha'\,dx$

Thus, we get: $\left|\overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f\alpha'\,dx\right| \leq M\varepsilon.$

Since this is true $\forall \varepsilon > 0$,

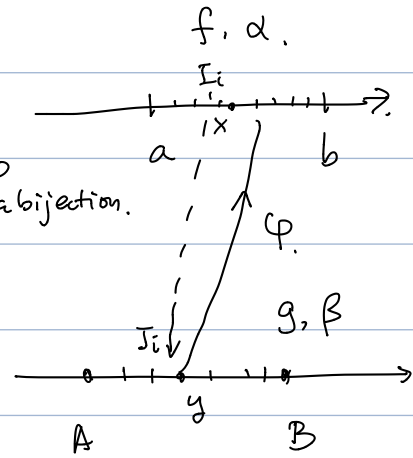
$$\Rightarrow \quad \overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\cdot\alpha'\cdot dx.$$

Hence $f \in R(\alpha)$ iff $f\alpha' \in R$, and $\int f\,d\alpha = \int f\alpha'\,dx$. #.

Thm 6.19 (change of variable).

Suppose α is increasing on $[a,b]$, and $f \in R(\alpha)$.

Suppose $\varphi : [A,B] \to [a,b]$ is a strictly increasing continuous function. $\Rightarrow \varphi$ is a bijection.



Define. $\beta(y) = \alpha(\varphi(y))$, and

$$g(y) = f(\varphi(y)).$$

Then. $g \in R(\beta)$, and

$$\int_A^B g \, d\beta = \int_a^b f \, d\alpha.$$

we are relabelling the interval $[a,b]$.

Pf: A partition $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[a,b]$ corresponds to a partition $Q = \{y_0 < \cdots < y_n\}$ of $[A,B]$.
Let I_i be the i-th interval in P, and J_i be the i-th interval in Q.

$$U(P,f,\alpha) = \sum_i \left(\sup_{x \in I_i} f(x)\right)(\alpha(I_i))$$

$$= \sum_i \left(\sup_{J_i} g(y)\right) \cdot (\beta(J_i)) = U(Q,g,\beta).$$

similarly $L(P,f,\alpha) = L(Q,g,\beta)$

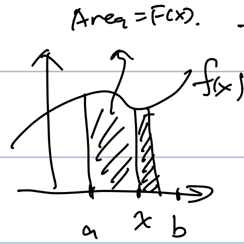
$$\Rightarrow \quad U(g,\beta) = L(g,\beta) = U(f,\alpha) = L(f,\alpha). \quad \#$$

- Relation between integration and differentiation.

(in this section, only talk about Riemann integral.) $\int f \, dx$.

Thm 6.20 Let $f \in \mathcal{R}$ on $[a,b]$. For $a \le x \le b$,

Let
$$F(x) = \int_a^x f(t)\cdot dt.$$



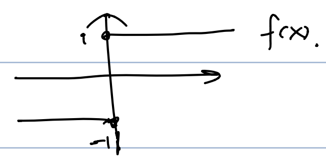
Thm 6.12. if $f \in \mathcal{R}$ on $[a,b]$ then $\forall\, [c,d] \subset [a,b]$ f is also integrable.

Then ① $F(x)$ is continuous on $[a,b]$.

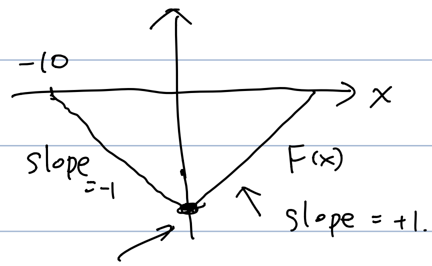
and ② if $f(x)$ is continuous at $x_0 \in [a,b]$, then $F(x)$ is differentiable at x_0, with $F'(x_0) = f(x_0)$.

Rmk: If $f(x)$ is not continuous at x_0, then it is possible $F(x)$ is not differentiable at x_0.

e.g. $$f(x) = \begin{cases} -1 & x \le 0 \\ 1 & x > 0. \end{cases}$$



$$F(x) = \int_{-10}^{x} f(t)\cdot dt$$



not differentiable at $x=0$

Pf: (1). Since $f \in \mathcal{R}$, hence f is bounded. Put $M = \sup_{[a,b]} |f(x)|$. If $x, y \in [a,b]$, $x < y$, then

$$F(y) - F(x) = \int_x^y f(t)\, dt.$$

$$\Rightarrow \quad |F(y) - F(x)| = \left| \int_x^y f(t)\, dt \right| \le \int_x^y |f(t)|\cdot dt$$
$$\le \int_x^y M\cdot dt = M\cdot |y-x|.$$

Hence F is Lipschitz continuous, with constant M.

(2) Suppose f is continuous at x_0. Hence for any $\varepsilon > 0$, $\exists\, \delta > 0$, s.t. $\forall\, x \in [a,b]$, $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.

Then $\forall\, s < t$ in $[a,b]$, s.t. $s \in (x_0 - \delta, x_0]$ and $t \in [x_0, x_0 - \delta)$, then

$$\frac{F(t) - F(s)}{t - s} = \frac{1}{t-s} \int_s^t f(u)\, du.$$

$$f(x_0) = \frac{1}{t-s} \int_s^t f(x_0) \cdot du.$$

$$\left| \frac{F(t) - F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{t-s} \int_s^t (f(u) - f(x_0))\, du \right|$$

$$\leq \frac{1}{t-s} \int_s^t |f(u) - f(x_0)|\, du.$$

$$\leq \frac{1}{t-s} \int_s^t \varepsilon \cdot du = \varepsilon.$$

Since this holds for all ε. (and corresponding δ ...), $F'(x_0) = f(x_0)$. #.

Thm (Fundamental Theorem of Calculus).

(wrong statement:) Let F be a differentiable function on $[a,b]$, let $f(x) = F'(x)$. assume f is bound. Then.

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

↗ what is wrong is that, f may not be Riemann integrable.
(Volterra function).

(Correct Statement) : Let $f \in \mathcal{R}$ on $[a,b]$. And assume.
$\exists$ differentiable function $F(x)$, on $[a,b]$, s.t. $F'(x) = f(x)$.
then.

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Fix $\varepsilon > 0$.

Pf : ∘ For any partition P,

$$F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1}).$$

• $\because f \in \mathcal{R}$. hence $\exists P$. s.t.

$$U(P,f) - L(P,f) < \varepsilon$$

. Then. $L(P,f) \leq \int_a^b f\, dx \leq U(P,f)$

and.

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \\ &\underset{\exists s_i \in I_i}{\overset{\text{MVT.}}{=}} \sum_{i=1}^{n} F'(s_i) \cdot \Delta x_i \\ &= \sum_{i=1}^{n} f(s_i) \cdot \Delta x_i \in [L(P,f), U(P,f)] \end{aligned}$$

• $\left| F(b) - F(a) - \int_a^b f\, dx \right| \leq U(P,f) - L(P,f) < \varepsilon.$

Since ε is arbitrary,

$$F(b) - F(a) = \int_a^b f \cdot dx. \qquad \#.$$

Thm. Suppose F and G. are differentiables and
F', G' are integrable. $f = F'$, $g = G'$.

Then.
$$\int_a^b F(x)\, g(x)\, dx = F(b)\, G(b) - F(a)\, G(a) - \int_a^b f(x)\, G(x) \cdot dx.$$

Pf: Let $H(x) = F(x)\, G(x)$, then H is differentiable.

$$H'(x) = F'(x) \cdot G(x) + F \cdot G'$$
$$= f \cdot G + F \cdot g.$$

$H'(x)$ is integrable.

($\because$ G is continuous $\Rightarrow$ G integrable.
f is integrable.
$\Rightarrow G \cdot f$ is integrable.)

Now we apply fundamental thm of calculus

$$\int_a^b H'(x)\, dx = H(b) - H(a).$$

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