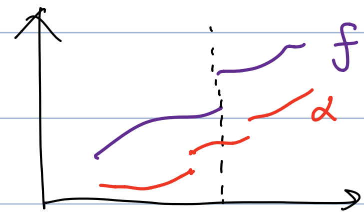
Last time: ① $f:[a,b]\to\mathbb{R}$ bounded ② $\alpha:[a,b]\to\mathbb{R}$ increasing.

Thm 6.8: if f is continuous, then $f\in\mathcal{R}(\alpha)$.

Thm 6.9: if f is monotonic and α is continuous, then $f\in\mathcal{R}(\alpha)$.

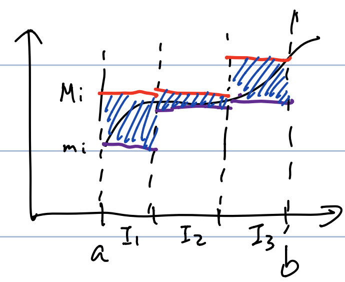
Today:

Thm 6.10: if f is discontinuous only at finitely many points, and α is continuous where f is discontinuous, then $f\in\mathcal{R}(\alpha)$.



idea: Given a partition P. $P=\{a=x_0<x_1<x_2<\cdots<x_n=b\}$.

$$U(P,f,\alpha)-L(P,f,\alpha)=\sum_{i=1}^{n}(M_i-m_i)\cdot\Delta\alpha_i$$



$U_P-L_P=$ "area" of the shaded region.

$I_i=[x_{i-1},x_i]$.

$\alpha(I_i)=\alpha(x_i)-\alpha(x_{i-1})$

we will separate the intervals in the partition into 2 type.

(A). ε $\boxed{}$ $\Delta\alpha_i$ height M_i-m_i is "under control" (good intervals)

(B) $\boxed{}$ $\Delta\alpha_i$ height M_i-m_i is not "under control" (bad intervals), we need to bound $\sum_{I_i \text{ bad}}\Delta\alpha_i$

(M_i-m_i) $\quad m=\inf f(x)\le m_i\le M_i\le\sup f(x)=M$.

$M_i-m_i\le M-m.$ ← a global bound.

or let $K=\sup|f(x)|$. then $M-m\le K-(-K)$
$=2K$.

Pf: Fix $\varepsilon > 0$. Let $E = \{c_1 < c_2 < \cdots < c_m\}$ be the set of discontinuities for f. (w.l.o.g., assume $E \subset (a,b)$).

Step 1: Since α is continuous at c_i, hence:

$$\alpha(c_i) = \lim_{t \to c_i^-} \alpha(t) = \lim_{t \to c_i^+} \alpha(t).$$

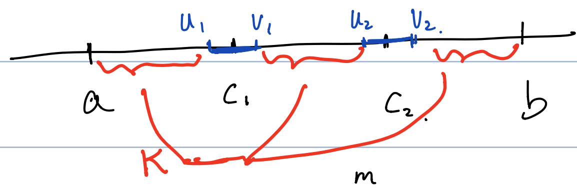
Hence we can take (u_i, v_i) around c_i, s.t.

$$\alpha(v_i) - \alpha(c_i) \leq \frac{\varepsilon}{2m}, \quad \alpha(c_i) - \alpha(u_i) \leq \frac{\varepsilon}{2m}.$$

$$\Rightarrow \quad \alpha(u_i) - \alpha(v_i) \leq \frac{\varepsilon}{m}.$$

$$\Rightarrow \quad \sum_{i=1}^{m} \alpha(u_i) - \alpha(v_i) \leq \varepsilon.$$

(bound of the total weights of the bad intervals)



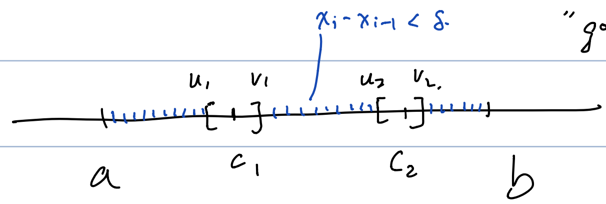
Step 2: Let $K = [a,b] \setminus \bigcup_{j=1}^{m} (u_i, v_i)$, K is a finite disjoint union of closed interval. Since f is continuous on K, and K is compact, hence f is uniformly continuous on K. Hence $\exists \delta > 0$. s.t. $\forall x, y \in K$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

step: Let P be a partition of $[a,b]$ satisfying

(1) $[u_i, v_i]$ are intervals in P. called "jump interval" or "bad int."

(2). If $I_i = [x_{i-1}, x_i]$ is not a jump interval, (i.e. $I_i \subset K$) "good interval"

then $|x_i - x_{i-1}| < \delta$.



$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta\alpha_i$$

$$= \sum_{I_i \text{ good}} (M_i - m_i) \cdot \Delta\alpha_i + \sum_{I_i : \text{bad}} (M_i - m_i) \cdot \Delta\alpha_i$$

$$\le \sum_{I_i \text{ good}} \varepsilon \cdot \Delta\alpha_i + \sum_{I_i \text{: bad.}} (M-m) \cdot \Delta\alpha_i$$

$$= \varepsilon \cdot \sum_{I_i\text{: good}} \Delta\alpha_i + (M-m) \sum_{I_i\text{: bad}} \Delta\alpha_i$$

$$\le \varepsilon \cdot [\alpha(b) - \alpha(a)] + (M-m) \cdot \varepsilon.$$

$$= \varepsilon \, [\underbrace{\alpha(b) - \alpha(a) + M - m}_{\text{a constant that only depends on } f \text{ and } \alpha.}]$$

Hence, by choosing $\varepsilon > 0$ small enough, and get corresponding partition, we can make $\mathcal{U}(P, f, \alpha) - L(P, f, \alpha)$ as small as we want. $\Rightarrow f \in R(\alpha)$ by Cauchy ~~conti~~ criteria. #.

Thm 6.11: Say: $f: [a,b] \to [m, M]$

and $\phi: [m, M] \to \mathbb{R}$ is continuous,

If f is integrable w.r.t. α. then $h = \phi \circ f$ is integrable. w.r.t. α.

Pf: Fix $\varepsilon > 0$.

- Since ϕ is uniformly continuous, hence $\exists \, \delta > 0$, s.t. $\delta < \varepsilon$ s.t. if $x, y \in [m, M]$ and $|x-y| < \delta$, then $|\phi(x) - \phi(y)| < \varepsilon$. Let $M^* = \sup \phi$, $m^* = \inf \phi$, i.e. $m^* \le \phi(x) \le M^*$. $\forall x \in [m, M]$.

- Since f is integrable w.r.t. α, $\exists \, P = \{\overset{a}{\overset{\shortparallel}{x_0}} < \cdots < \overset{b}{\overset{\shortparallel}{x_n}}\}$ partition of $[a,b]$, s.t.

$$\mathcal{U}(P, f, \alpha) - L(P, f, \alpha) < \delta^2$$

Let $M_i = \sup_{I_i} f(x)$, $m_i = \inf_{I_i} f(x)$, $I_i = [x_{i-1}, x_i]$,

$M_i^* = \sup_{I_i} h(x)$, $m_i^* = \inf_{I_i} h(x)$, $h(x) = \phi(f(x))$.

We say I_i is a "good interval", if $M_i - m_i < \delta$

and. I_i is a "bad interval", if $M_i - m_i \geq \delta$.

$$\delta^2 \geq U(P,f,\alpha) - L(P,f,\alpha) \geq \sum_{I_i:\text{ bad}} (M_i - m_i)\cdot \Delta\alpha_i$$

$$\geq \sum_{I_i\text{ bad}} \cdot\, \delta \cdot \Delta\alpha_i = \delta \cdot \sum_{I_i\text{ bad}} \Delta\alpha_i$$

$$\Rightarrow \boxed{\sum_{I_i\text{ bad}} \Delta\alpha_i \leq \delta.}$$

- If I_i is good, then $M_i - m_i < \delta$, then $M_i^* - m_i^* < \varepsilon$.

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{\text{good}} (M_i^* - m_i^*)\cdot \Delta\alpha_i + \sum_{\text{bad}} (M_i^* - m_i^*)\cdot \Delta\alpha_i$$

$$\leq \varepsilon \cdot \sum_{\text{good}} \Delta\alpha_i + (M^* - m^*) \cdot \sum_{\text{bad}} \Delta\alpha_i$$

$$\leq \varepsilon \cdot [\alpha(b) - \alpha(a)] + (M^* - m^*)\cdot \delta \qquad \text{since } (\delta \leq \varepsilon)$$

$$\leq \varepsilon\,[\alpha(b) - \alpha(a) + M^* - m^*].$$

$---- \Rightarrow h$ is integrable w.r.t α. #.

Property: "The integration operation $\int f\, d\alpha$ is linear in both f and α."

(Recall: $F: V \to W$ is a linear function

vector space.

if $\forall\, v_1, v_2 \in V$, $c \in \mathbb{R}$. we have

$$\left.\begin{array}{rcl} F(v_1+v_2) &=& F(v_1)+F(v_2) \\ F(cv_1) &=& c\cdot F(v_1). \end{array}\right\}$$

"Linear in f"

Thm 6.12 : ① If $f_1, f_2 \in \mathcal{R}(\alpha)$, and $c \in \mathbb{R}$. Then.

- $f_1+f_2 \in \mathcal{R}(\alpha)$, $\int f_1+f_2\, d\alpha = \int f_1\, d\alpha + \int f_2\, d\alpha$.
- $c\cdot f_1 \in \mathcal{R}(\alpha)$, $\int c\cdot f_1\, d\alpha = c\cdot \int f_1\, d\alpha$.

② Linearity in α is similar.

③ If $f, g \in \mathcal{R}(\alpha)$, and $f(x) \leq g(x)$. $\forall x \in [a,b]$.

then $\int f\, d\alpha \leq \int g\, d\alpha$. ..

Thm 6.13 ① If $f, g \in \mathcal{R}(\alpha)$, then $f\cdot g \in \mathcal{R}(\alpha)$.

② If $f \in \mathcal{R}(\alpha)$, then $|f| \in \mathcal{R}(\alpha)$.

and. $\left|\int f\cdot d\alpha\right| \leq \int |f|\cdot d\alpha$.

Pf: ①. We first prove that $f \in \mathcal{R}(\alpha) \Rightarrow f^2 \in \mathcal{R}(\alpha)$.

let $\phi(y) = y^2$, then $f^2(x) = \phi(f(x))$. hence

integrable w.r.t. α. by Thm 6.11.

Now using the "polarization trick":

$f\cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$ RHS is integrable.

$\therefore f\cdot g$ is integrable.

② . Let $\phi(y) = |y|$, hence $|f|(x) = \phi(f(x))$, is integrable.

. since $f(x) \leq |f|(x)$ $-f(x) \leq f(x)$.

hence $\pm \int f\, d\alpha \leq \int |f| \cdot d\alpha$.

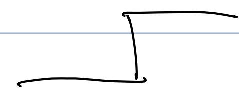
$\therefore \left|\int f\, d\alpha\right| = \max \left\{ \int f\, d\alpha, -\int f\, d\alpha \right\} \leq \int |f|\, d\alpha$. #.

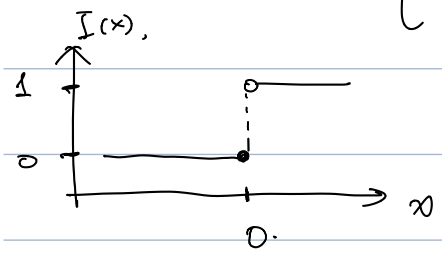
Special Case

- α is a sum of step functions.

Step function:

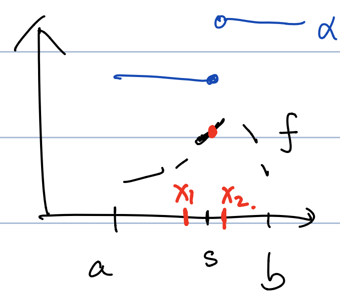
$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1. & x > 0 \end{cases}$$





Thm 6.15: If $f : [a,b] \to \mathbb{R}$, and is continuous at $s \in (a,b)$, and $\alpha(x) = I(x-s)$, then

$$\int f \cdot d\alpha = f(s).$$



Pf: Consider partition. $P = \{ \underset{\overset{\|}{a}}{x_0} < x_1 < x_2 < \underset{\overset{\|}{b}}{x_3} \}$.

$$\mathcal{U}(P, f, \alpha) = \left(\sup_{[x_1, x_2]} f(x) \right) \cdot 1$$

$$L(P, f, \alpha) = \left(\inf_{[x_1, x_2]} f(x) \right) \cdot 1.$$

Hence, by continuity of f at $x = s$, as $x_2 - x_1 \to 0$, both $U_P, L_P \to f(s)$. #.

Rmk: "$d\alpha = \delta(x-s)\, dx$". in this case.

Thm 6.16. Let $c_n \geq 0$ for $n = 1, 2, \cdots$, $\sum c_n < \infty$. let $\{s_n\}$ be a sequence of distinct points in $[a,b]$.

$$\alpha(x) = \sum_{n=1}^{\infty} c_n \cdot I(x - s_n) = \sum_{n:\, x > s_n} c_n.$$

Let $f : [a,b] \to \mathbb{R}$ be continuous. Then.

$$\int f \, d\alpha = \sum_{n=1}^{\infty} c_n \cdot f(s_n)$$

① α is indeed an increasing function on $[a,b]$.

② $f \in R(\alpha)$. Hence $\int f d\alpha$ exist

③. $\because m \leq f \leq M$

$m \cdot \sum c_n \leq \sum c_n \cdot f(s_n) \leq M \cdot \sum c_n$

using Cauchy condition.

$\Rightarrow \sum c_n f(s_n)$ exist.

Pf: Let $\varepsilon > 0$. Choose N, such that

$$\sum_{N+1}^{\infty} c_n < \varepsilon.$$

Let $\alpha(x) = \alpha_1(x) + \alpha_2(x)$, where

$\alpha_1(x) = \sum_1^N c_n \cdot I(x - s_n)$

$\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n \cdot I(x - s_n)$.

$0 \leq \alpha_2(x) \leq \varepsilon$

- $$\int f \cdot d\alpha_1 = \int f \, d\left(\sum_{i=1}^N c_n \cdot I(x - s_n)\right)$$
$$= \sum_{i=1}^N c_n \cdot \int f \cdot d(I(x - s_n))$$
$$= \sum_{i=1}^N c_n \cdot f(s_n).$$

- $$\left|\int f \, d\alpha_2\right| \leq \int |f| \cdot d\alpha_2 \leq \sup |f| \cdot \int d\alpha_2$$

$\int d\alpha_2 = \alpha_2(b) - \alpha_2(a)$.

$$= \big(\underbrace{\sup |f|}_{=M}\big) \cdot \varepsilon.$$

$$\left|\int f \cdot d\alpha - \sum_{i=1}^{N} c_n \cdot f(s_n)\right| = \left|\int f \, d\alpha_2\right| \leq M \cdot \varepsilon$$

As $\varepsilon \to 0$, $N \to \infty$, and ~~this~~ hence

$$\lim_{N \to \infty} \sum_{i=1}^{N} c_n f(s_n) = \int f \, d\alpha. \qquad \#.$$

In the morning, we did an example.:

$\alpha(x) = I(x)$

$f(x) = I(x)$.

$\Rightarrow f$ is NOT integrable w.r.t. α.

relevant for Thm 6.10.
