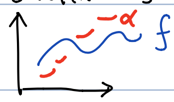
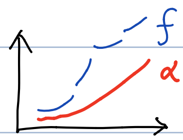
Last Time:

Thm. 6.8 Rudin: If f is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

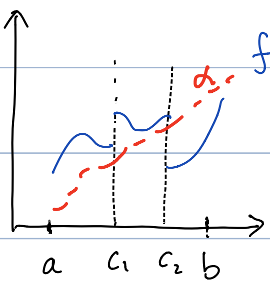


Thm 6.9 Rudin: If f is monotonic, and if α is continuous (and also monotone increasing). then $f \in \mathcal{R}(\alpha)$.



Today:

Thm 6.10: If $f: [a,b] \to \mathbb{R}$ is bound, and has finitely many discontinuities. And if α is continuous when f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.

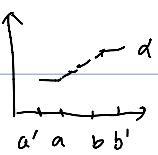


Ex: f discontinous at $x = c_1$, $x = c_2$

α is continuous at c_1, c_2.

Pf: Let $\varepsilon > 0$. be given. Put $M = \sup |f(x)|$. Let E be the set of points where f is discontinuous. By assumption, $E = \{c_1, c_2, \dots, c_m\}$ is a finite set.

[W.l.o.g. assume $E \subset (a,b)$. If not, we can. extend the interval $[a,b]$ to a larger interval $[a', b']$, $a' < a$, $b' > b$. Set $\alpha(x) = \alpha(a)$ if $x < a$, $\alpha(x) = \alpha(b)$ if $x > b$.



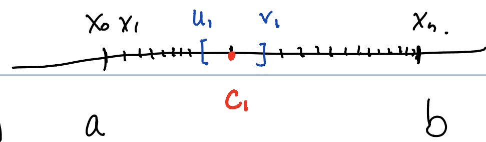
set $f(x) = 0$ if $a' \le x \le a$, or $b < x \le b'$.]

Step 1: choose small enough intervals $[u_j, v_j]$ around c_j, such that $\sum_{j=1}^{m} \alpha(v_j) - \alpha(u_j) < \varepsilon$., and $[u_j, v_j]$ are disjoint. We can achieve this since α is continuous at c_j. , $\lim_{t\to c_j^+}\alpha(t) = \lim_{t\to c_j^-}\alpha(t) = \alpha(c_j)$

Step 2: Let $K = [a,b] \setminus \bigcup_{j=1}^{m} (u_j, v_j)$. K is compact, and f is continuous on K, hence f is uniformly continuous. For the given choice of $\varepsilon > 0$, $\exists\, \delta > 0$, s.t. if $x, y \in K$, $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Step 3: Let P be a partition of $[a,b]$ of the following form: ① $\{u_i, v_i\}_{i=1}^{m}$ are in P.

and there are no other points in P lying in $[u_i, v_i]$



② if $[x_{i-1}, x_i]$ is not in $[u_j, v_j]$ for some j, then $|x_i - x_{i-1}| < \delta$.

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta\alpha_i .$$

- if the interval $[x_{i-1}, x_i]$ "I_i" is not a "jump interval", ie. contained in K, then $M_i - m_i < \varepsilon$, thanks to the uniform continuity of f on K.
- if the interval $[x_{i-1}, x_i]$ is a jump interval $[u_j, v_j]$, then $M_i - m_i \le 2M$, ($\sup |f(x)| = M \Rightarrow -M \le f(x) \le M$
  $\Rightarrow -M \le \sup_{I_i} f(x) \le M$, $-M \le \inf_{I_i} f(x) \le M$ $\Rightarrow -M \le M_i \le M$, $-M \le m_i \le M$.)

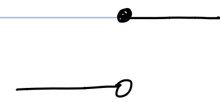
$$\le \varepsilon \cdot \sum_{\substack{i=1 \\ I_i \text{ not jump interval}}}^{n} \Delta\alpha_i + 2M \cdot \sum_{\substack{i \\ I_i \text{ is a jump interval}}} \Delta\alpha_i$$

$$\le \varepsilon \cdot [\alpha(b) - \alpha(a)] + 2M \cdot \varepsilon .$$

$$= \varepsilon \cdot [\alpha(b) - \alpha(a) + 2M].$$

Since $\varepsilon > 0$ is arbitrary, we can achieve the Cauchy condition. #

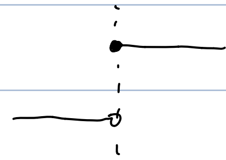
Rmk: If f and α are both discontinuous at $x=c$, then it is possible that $f \notin \mathcal{R}(\alpha)$.



α.

Ex: $[a,b] = [-1,1]$, $\quad \alpha(x) = \begin{cases} 0 & x<0 \\ 1 & x \geq 0 \end{cases}$

$$f(x) = \begin{cases} 0 & x<0 \\ 1 & x \geq 0. \end{cases}$$

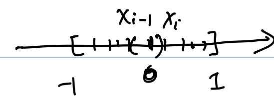


f.

Let P be any partition of $[-1,1]$. There are 2 cases

(1) if P contains an interval $I_i = [x_{i-1}, x_i]$, s.t. $0 \in (x_{i-1}, x_i)$, then.

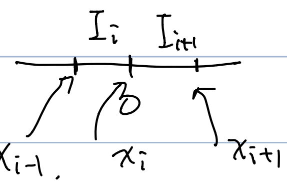
$$M_i = \sup_{x\in I_i} f(x) = 1, \qquad m_i = \inf_{I_i} f(x) = 0.$$



$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = 1.$$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{j=1}^{n} (M_j - m_j)\cdot\Delta\alpha_j = (1-0)\cdot 1 = 1.$$

(2) if P has $x=0$ as one of the cut points, i.e. $x_i = 0$ for some i. Then.



$$\begin{cases} M_i = \sup_{x\in I_i} f(x) = 1 & (\because f(x_i)=1). \\ m_i = \inf_{x\in I_i} f(x) = 0 \end{cases}$$

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) = 1$$

$$\begin{cases} M_{i+1} = 1 \\ m_{i+1} = 1. \end{cases}$$

$$\Delta\alpha_{i+1} = \alpha(x_{i+1}) - \alpha(x_i) = 0.$$

I_i $\qquad$ I_{i+1}

$$U(P,f,\alpha) - L(P,f,\alpha) = (1-0)\cdot 1 + (1-1)\cdot 0. = 1.$$

$\Rightarrow \forall P$ partition, $\mathcal{U}(P) - L(P) = 1$.

$\therefore f$ is not integrable.

<u>Thm 6.11</u>. If $f: [a,b] \to \mathbb{R}$ is integrable w.r.t. α.

and assume $f(x) \in [m, M]$ $\forall x \in [a,b]$. If

$\phi: [m, M] \to \mathbb{R}$ is continuous. Then.

$$h(x) = \phi(f(x))$$

is integrable w.r.t. α.

<u>Pf</u>: Fix an $\varepsilon > 0$. Since ϕ is unif. continuous on $[m, M]$,

$\exists\ \delta > 0$. s.t. if $|y_1 - y_2| < \delta$, then $|\phi(y_1) - \phi(y_2)| < \varepsilon$.

Since f is integrable, $\exists\ P$ partition, s.t.

$$\mathcal{U}(P, f, \alpha) - L(P, f, \alpha) < \delta^2.$$

For interval $I_i = [x_{i-1}, x_i]$, let M_i, m_i be sup / inf of f; and let M_i^*, m_i^* be sup or inf for $h(x)$.

We say $i \in A$, if $M_i - m_i < \delta$; otherwize we say $i \in B$.

$\{1, \cdots, n\} = A \sqcup B$.

$$h(x) = \phi(\underline{f(x)})$$

For $i \in A$, since $M_i - m_i < \delta$, hence $M_i^* - m_i^* < \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \leq 2 \cdot \sup |\phi(x)| = 2K$.

$$\delta \cdot \sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i) \cdot \Delta\alpha_i \leq \mathcal{U}(P, f, \alpha) - L(P, f, \alpha) < \delta^2.$$

$$\Rightarrow \quad \sum_{i \in B} \Delta \alpha_i < \delta.$$

Finally, $\mathcal{U}(P,h,\alpha) - L(P,h,\alpha) = \sum_{i=1}^{n} (M_i^* - m_i^*) \cdot \Delta\alpha_i$

$$= \sum_{i \in A} (M_i^* - m_i^*) \cdot \Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \cdot \Delta\alpha_i$$

$$\leq \sum_{i \in A} \varepsilon \cdot \Delta\alpha_i + \sum_{i \in B} 2K \cdot \Delta\alpha_i$$

$$\leq \varepsilon \cdot [\alpha(b) - \alpha(a)] + 2K \cdot \delta.$$ ← (when we choose δ based on ε, we can make furthere requirement that $\delta < \varepsilon$.)

$$\leq \varepsilon \, [\alpha(b) - \alpha(a) + 2K].$$

Since $\varepsilon > 0$ is arbitrary, $\exists P$, s.t. $\mathcal{U}(P,h,\alpha) - L(P,h,\alpha)$ can be made smaller than any given positive number. #.

<u>Properties of integral</u>:

<u>Thm 6.12</u>: " $\int f \, d\alpha$ is linear in f and α ".

- If $f_1, f_2 \in R(\alpha)$, then. $f_1 + f_2 \in R(\alpha)$

$$\int (f_1 + f_2) \, d\alpha = \int f_1 \, d\alpha + \int f_2 \, d\alpha$$

$$\int c f_1 \, d\alpha = c \cdot \int f \, d\alpha. \qquad c \in \mathbb{R}$$

- If $f \in R(\alpha_1)$, $f \in R(\alpha_2)$, then $f \in R(\alpha_1 + \alpha_2)$.

$$\int f \, d(\alpha_1 + \alpha_2) = \int f \, d\alpha_1 + \int f \, d\alpha_2$$

$$\int f \, d(c \cdot \alpha_1) = c \cdot \int f \, d\alpha_1 \qquad \forall c \in \mathbb{R}.$$

<u>Warning</u>: if $f_1 \in R(\alpha_1)$ and $f_2 \in R(\alpha_2)$, then $f_1 + f_2$ may not be in $R(\alpha_1 + \alpha_2)$.

Thm 6.13. (1) If $f, g \in R(\alpha)$, then $f \cdot g \in R(\alpha)$.

(2). If f is integrable, then $|f|$ is integrable.

$$\left| \int f \, d\alpha \right| \leq \int |f| \cdot d\alpha.$$

(apply Thm 6.11).