Recap:

- partition P of $[a,b]$. $a,b \in \mathbb{R}$.

  $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$. $P = \{x_0, x_1, \cdots, x_n\} \subset [a,b]$.

(new): - refinement of partition: Q is a refinement of P, if $Q \supset P$ as subsets of $[a,b]$

- Given 2 partitions, P_1, P_2.

  $P_1 \cup P_2$ is a partition, common refinement of P_1 and P_2.

cumulative weight function.

- $\alpha : [a,b] \to \mathbb{R}$ increasing function.

  meaning: $\forall x, y \in [a,b]$, $x < y$. $\alpha(y) - \alpha(x)$ is "the weight" in the interval $[x, y)$.

  (open or closed boundary is only by convention)

integrand

- $f : [a,b] \to \mathbb{R}$ bounded.

$I_i = [x_{i-1}, x_i]$

Given P, α, f. we define $P = \{x_0, \cdots, x_n\}$

$$\mathcal{U}(P, f, \alpha) = \sum_{i=1}^{n} M_i \cdot \Delta\alpha_i \qquad \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \cdot \Delta\alpha_i \qquad M_i = \sup \{f(x) \mid [x_{i-1}, x_i]\}$$

$\because M_i \geq m_i \quad \therefore \mathcal{U}(P, f, \alpha) \geq L(P, f, \alpha)$. $\qquad m_i = \inf_{I_i} \{f(x) |$

$$\mathcal{U}(f, \alpha) = \inf \{\mathcal{U}(P, f, \alpha) \mid P \text{ partition of } [a,b]\}$$

$$L(f, \alpha) = \sup \{L(P, f, \alpha) \mid \cdots\}.$$

Def: f is integrable w.r.t weight $\alpha(x)$ on $[a,b]$, if $\mathcal{U}(f, \alpha) = L(f, \alpha)$. We denote the common value

$$\int_a^b f \, d\alpha \quad \text{or.} \quad \int_a^b f(x) \, d\alpha(x)$$

And write $f \in \mathcal{R}(\alpha)$ on $[a,b]$

the set of Riemann-Stieltjes integrable fcn.

$\underline{Q_1}$: how do we know that

$$L(f,\alpha) \leq U(f,\alpha) \; ?$$

$$1 \leq 2$$

$$3 \leq 4$$

$$\max\{1,3\} \; \cancel{\leq} \; \min\{2,4\}$$

$\underline{\text{Lemma}}$: If Q (a partition) is a refinement of P (a partition) on $[a,b]$.

Then. "the approximate integrals bounds get better", i.e.

$$\underset{\substack{\shortparallel \\ L(P,f,\alpha)}}{L_P} \leq L_Q \leq U_Q \leq U_P.$$

$\underline{\text{Pf}}$: Consider only the case that Q has one more "cut point" than P. Say the point is x^*, and $x_{i-1} < x^* < x_i$, where $P = \{x_0, x_1, \cdots, x_n\}$. Then.

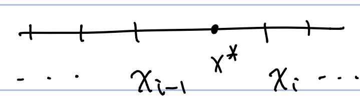
$I_i = [x_{i-1}, x_i]$

$I_i^- = [x_{i-1}, x^*]$

$I_i^+ = [x^*, x_i]$

$$U_P - U_Q = \Big[\underbrace{\sup_{I_i} f(x)}_{M_i}\Big] \cdot \alpha(I_i)$$

$$- \Big(\underbrace{\sup_{I_i^-} f(x)}_{M_i^-}\Big) \cdot \alpha(I_i^-) - \Big(\underbrace{\sup_{I_i^+} f(x)}_{M_i^+}\Big) \alpha(I_i^+).$$



$\because \; \alpha(I_i) = \alpha(I_i^+) + \alpha(I_i^-)$

$\alpha(I_i) = \alpha(x_i) - \alpha(x_{i-1})$

$$\sup_{I_i} f(x) \geq \sup_{I_i^{\pm}} f(x), \quad M_i \geq M_i^+, \quad M_i \geq M_i^-$$

$$\therefore \; U_P - U_Q = (M_i - M_i^+) \cdot \alpha(I_i^+) + (M_i - M_i^-) \cdot \alpha(I_i^-)$$

$$\geq 0.$$

#.

(6.5 Rudin).

Thm: $L(f,\alpha) \leq \mathcal{U}(f,\alpha)$.

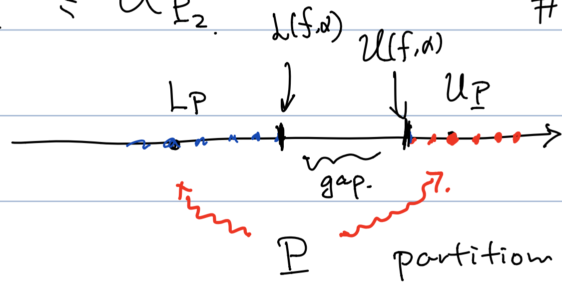
Pf: suffice to show that $\forall$ P_1, P_2 partitions,

$$L(P_1, f, \alpha) \leq \mathcal{U}(P_2, f, \alpha). \quad (*)$$

since if this is true, then apply $\sup_{P_1}$ $\inf_{P_2}$, we get the desired inequality

(*) follows by consider the common refinement $P_1 \cup P_2$.

$$L_{P_1} \overset{\text{by Lemma}}{\leq} L_{P_1 \cup P_2} \leq \mathcal{U}_{P_1 \cup P_2} \overset{\text{by Lemma.}}{\leq} \mathcal{U}_{P_2}. \quad \#$$



Thm (Cauchy Condition)

f is integrable w.r.t. α

$\iff$ $\forall \varepsilon > 0$, $\exists$ P partition, such that

$$\mathcal{U}_P - L_P < \varepsilon.$$

Pf: $\Leftarrow$ $\because$ for any P partition,

$$L_P \leq L \leq \mathcal{U} \leq \mathcal{U}_P,$$

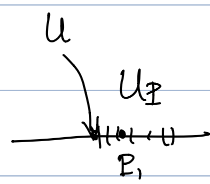
$\therefore$ $0 \leq \mathcal{U} - L \leq \mathcal{U}_P - L_P$

$\because$ $\forall \varepsilon > 0$, $\exists P$, s.t. $\mathcal{U}_P - L_P < \varepsilon$.

$\therefore$ $\forall \varepsilon > 0$, $0 \leq \mathcal{U} - L < \varepsilon$ $\quad\therefore$ $\mathcal{U} = L$.

$\Rightarrow$ $\because$ $\mathcal{U} = \inf\{\mathcal{U}_P \mid P \text{ partition}\}$

$\therefore$ $\exists P_1$, s.t. $\mathcal{U}_{P_1} - \mathcal{U} < \frac{\varepsilon}{2}$

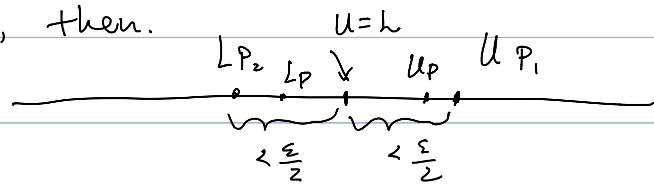


similarly

$\because \quad L = \sup\{ L_P \mid P \text{ partition} \}$.

$\exists P_2$, s.t. $\quad L - L_{P_2} < \frac{\varepsilon}{2}$.

Let $P = P_1 \cup P_2$, then.



$\therefore \quad U_P - L_P \leq U_{P_1} - L_{P_2} < \varepsilon$. #.

History (from Ross).

- The original definition by Riemann (German. (1826–1866)) is using "Riemann sum"

Given a partition P, choose "sample points"

$$s_i \in I_i = [x_{i-1}, x_i].$$

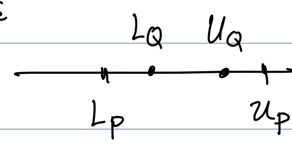
$$L(P,f) \leq \sum_{i=1}^{n} f(s_i) \cdot \Delta x_i \leq U(P,f)$$

- introduced by Darboux (French. 1842 – 1917)
paper 1875.

- Read Ross for the equivalence between Rieman's approach and Darboux's approach.

Thm (6.7 Rudin): Fix f, α.

(1) If partition P satisfies. $U_P - L_P < \varepsilon$, then any refinement Q satisfies $U_Q - L_Q < \varepsilon$



(2) If $U_P - L_P < \varepsilon$, and $\{s_i \in I_i\}$ and $\{t_i \in I_i\}$ are 2 sets of sample points, then.

$$\sum_{i=1}^{n} |f(t_i) - f(s_i)| \cdot \Delta\alpha_i < \varepsilon.$$

Pf : $\because \ |f(t_i) - f(s_i)| \leq M_i - m_i$

$\therefore \ \sum |f(t_i) - f(s_i)| \cdot \Delta\alpha_i \leq \sum (M_i - m_i) \cdot \Delta\alpha_i = U_P - L_P < \varepsilon.$

(3). If $U_P - L_P < \varepsilon$, and f is integrable, and $\{s_i \in I_i\}$

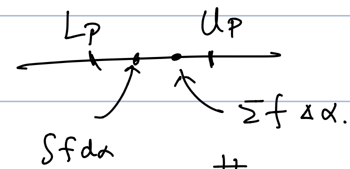
Then

$$\left| \int_a^b f\, d\alpha - \sum f(s_i) \cdot \Delta\alpha_i \right| < \varepsilon. \qquad (*)$$

Pf :

$$L_P < \int_a^b f\, d\alpha < U_P$$

$$L_P < \sum f(s_i) \cdot \Delta\alpha_i < U_P.$$



$\therefore$ $(*)$ holds.

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Thm 6.8 : If f is continuous on $[a,b]$, then f is integrable w.r.t. α on $[a,b]$.

Pf : • We use Cauchy condition to prove $f \in \mathcal{R}(\alpha)$. Namely $\forall \varepsilon > 0$, we need to show that. $\exists P$ partition of $[a,b]$.) s.t.

$$U_P - L_P < \varepsilon.$$

$$U_P - L_P = \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta\alpha_i$$

$M_i = \sup_{I_i} f(x) = \max_{I_i} f(x) = f(a_i)$ $\because I_i = [x_{i-1}, x_i]$ is compact for some $a_i \in I_i$.

$m_i = \inf_{I_i} (f(x)) = \min_{I_i} f(x) = f(b_i)$ for some $b_i \in I_i$.

$\because f$ is continuous on $[a,b]$, a compact set

$\therefore f$ is uniformly ~~co~~ continuous. , i.e. $\forall \eta > 0$. $\exists \delta > 0$.

such that $\forall x, y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \eta$.

Fix $\eta = \dfrac{\varepsilon}{\alpha(b) - \alpha(a)}$, (assuming denominator $\neq 0$), and choose corresponding δ.

Then, choose a partition P, s.t. $\Delta x_i < \delta$.

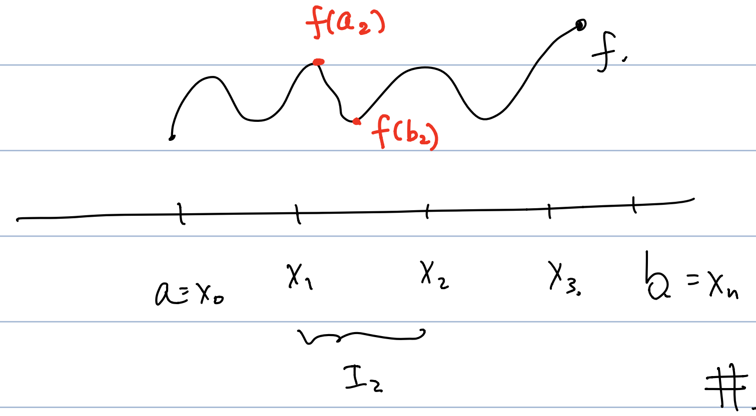
$$\mathcal{U}_P - L_P = \sum_i (M_i - m_i) \cdot \Delta \alpha_i = \sum_i [f(a_i) - f(b_i)] \cdot \Delta \alpha_i$$

$$\leq \sum_i \eta \cdot \Delta \alpha_i$$

$$= \eta \cdot \sum_i \Delta \alpha_i$$

$$= \eta \cdot [\alpha(b) - \alpha(a)]$$

$$= \varepsilon.$$



#.

<u>Thm 6.9 Rudin</u>: If f is monotonic and α is continuous.

then $f \in \mathcal{R}(\alpha)$.

<u>Pf</u>: . $\forall \varepsilon > 0$, need P s.t. $\mathcal{U}_P - L_P < \varepsilon$.

$\forall n > 0$ integer.

<u>Step 1</u>: construct "evenly distributed weight partition" P_n.

By continuity of $\alpha(x)$., for each $\alpha(a) \leq \mu \leq \alpha(b)$

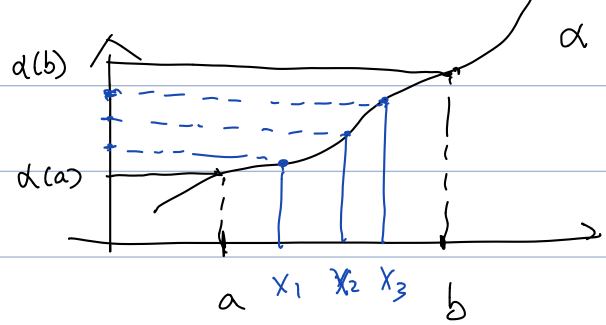
$\exists x_\mu$ s.t. $\alpha(x_\mu) = \mu$. (by I.V.T).

For each $n > 0$ integer, and $k = 1, \cdots, n-1$,

let $\mu_{n,k} = [\alpha(b) - \alpha(a)] \cdot \frac{k}{n} + \alpha(a)$, we get $x_1, \cdots, x_{n-1}$.

s.t. $\alpha(x_k) = \mu_{n,k}$. This gives us a partition

P_n, s.t. $\underline{\Delta \alpha(I_k) = \frac{1}{n} \cdot (\alpha(b) - \alpha(a))}$.



n=4.

Step2 :

$$U_{P_n} - L_{P_n} = \sum_i (M_i - m_i) \cdot \Delta\alpha_i$$

$$= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \cdot \frac{\alpha(b) - \alpha(a)}{n}$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \cdot \left(\sum_{i=1}^{n} f(x_i) - f(x_{i-1}) \right)$$

$$= \frac{[\alpha(b) - \alpha(a)]}{n} [f(b) - f(a)].$$

for n large enough, the above is less than ε.

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Ex (non integrable function) :

$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \in \mathbb{Q}^c \end{cases}$$

say on $[0,1]$, $\int f\, dx$.

$\cdot\ \forall P$ partition. $M_i = 1$, $m_i = 0$.

on interval $[0,1]$:

$$U(P,f) = \sum M_i \cdot \Delta x_i = 1 \cdot \sum \Delta x_i = 1 \cdot 1$$

$$L(P,f) = \sum m_i \ \Delta x_i = 0 \cdot \sum \Delta x_i = 0 \cdot 1$$

This function is integrable as Lebesgue integral.

$\because$ $f = 0$ on $\mathbb{Q}$, which has measure zero.

$\therefore$ $f \overset{a.e.}{=} 1$ on $\mathbb{R}$ "almost everywhere" (i.e. outside a measure zero set)