Last time: Definition of Riemann integral.

- Def: (partition of a closed bounded interval $[a,b]$).

$$P = \{ a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b \}.$$ n segments

$$\Delta x_i = x_i - x_{i-1}$$

- f: real and bounded fcn on $[a,b]$.

- $U(P,f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i \qquad M_i = \sup\{ f(x) \mid x \in [x_{i-1}, x_i] \}$

  $L(P,f) = \sum_{i=1}^{n} m_i \cdot \Delta x_i \qquad m_i = \inf\{ f(x) \mid x \in [x_{i-1}, x_i] \}.$

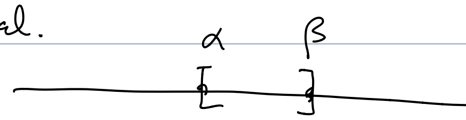
- $U(f) := \inf\{ U(P,f) \mid P \text{ partition of } [a,b] \}.$

  $L(f) := \sup\{ L(P,f) \mid P \text{ partition of } [a,b] \}.$

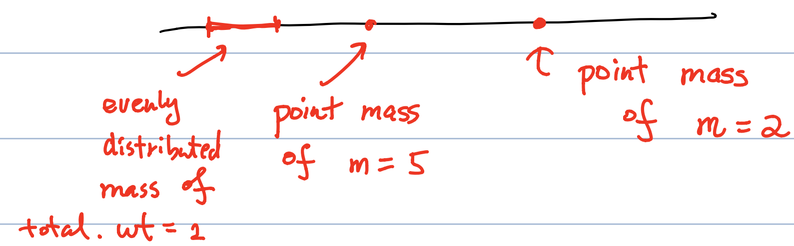
Today: Stieltjes integral.

Intuition: Riemann integral, for an interval $[\alpha, \beta]$, we use $|\beta - \alpha|$ for its length.

- another interpretation: let "weight" be distributed evenly on $\mathbb{R}$, then $|\beta - \alpha|$ is the amount of weight on this interval.



- To describe other type of weight distribution, such as point mass, or weights that are distribution

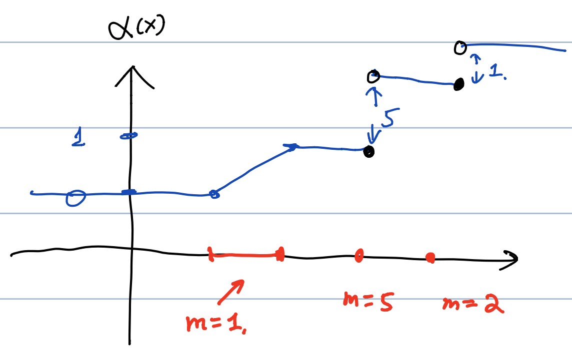


How to describe such a weight distribution?

Introduce a function $\alpha(x) =$ " weight lying on the interval $(-\infty, x)$ "

$(-\infty, x]$

In the above example,



using our convention

$\alpha(x)$ is left continuous in this example.

(it is not necessary so in general).

$\Rightarrow$ · $\alpha(x)$ is a monotone increasing function (if $x < y$ then $\alpha(x) \le \alpha(y)$)

Formal definition:

- Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing function.
- P partition, $f : [a,b] \to \mathbb{R}$ bounded.

$$U(P, f, \alpha) := \sum_{i=1}^{n} M_i \cdot \Delta\alpha_i .$$

$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

" the weight on the interval $[x_i, x_{i-1})$. "

$$L(P, f, \alpha) := \sum_{i=1}^{n} m_i \cdot \Delta\alpha_i$$

· $U(f, \alpha)$ and $L(f, \alpha)$ defined similarly.

<u>Def</u>: If $U(f,\alpha) = L(f,\alpha)$, we say f is integrable ~~wrt~~ with respect to α. And write $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

<u>Rmk</u>: (1) if $\forall x \in [a, b]$, $m \leq f(x) \leq M$. , then.

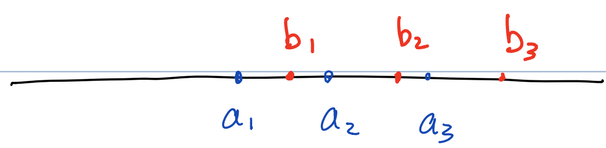
$$m(\alpha(b) - \alpha(a)) \leq \underbrace{\sum_{i=1}^{n} m_i \, \Delta\alpha_i}_{= L(P, f, \alpha)} \leq \underbrace{\sum_{i=1}^{n} M_i \cdot \Delta\alpha_i}_{= U(P, f, \alpha)} \leq M \cdot (\alpha(b) - \alpha(a))$$

(2) we want to show $L(f, \alpha) \leq U(f, \alpha)$.

note: if $\forall n \in \mathbb{N}$, $a_n \leq b_n$.

we cannot conclude that

$$\sup(a_n) \leq \inf(b_n).$$



<u>Def</u>: Let P, Q be 2 partitions of $[a, b]$.

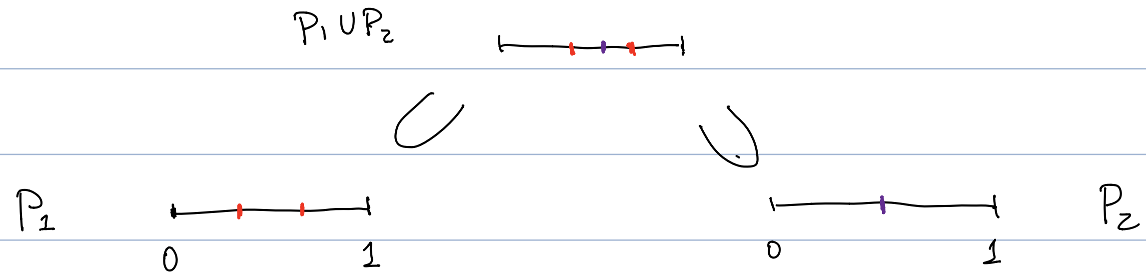
$P = \{ a = x_0 < x_1 < x_2 < \cdots < x_n = b \}$

$Q = \{ a = y_0 < y_1 < \cdots < y_m = b \}$.

So, P, Q can be identified as a finite subset of $[a, b]$. We say Q is a refinement of P, if $Q \supset P$ as subsets of $[a, b]$.

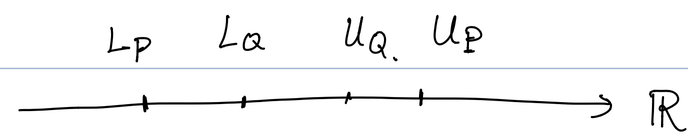
Given 2 partitions P_1 and P_2, we let $P_1 \cup P_2$ to denote their common refinement.

Ex



Lemma: If Q is a refinement of P, then

$$L(P, f, \alpha) \leq L(Q, f, \alpha)$$
$$U(P, f, \alpha) \geq U(Q, f, \alpha).$$

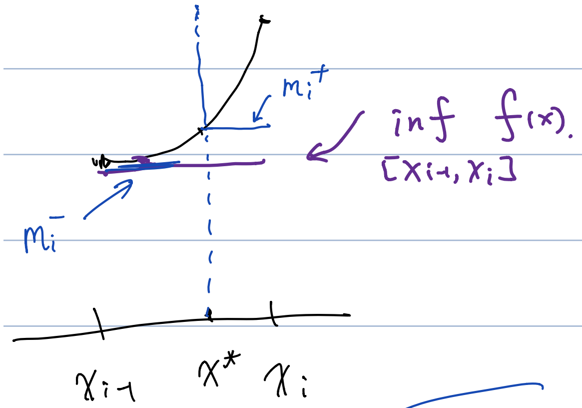


" refinement of partition improves the approximation "

Pf: Suffice to prove the case that, Q has one more point than P. Say the point is x^*.

$$Q = x_0 < x_1 < \cdots < \underline{x_{i-1} < x^* < x_i} < \cdots < x_n$$
$$P = x_0 < x_1 < \cdots < \underline{x_{i-1} < x_i} < \cdots x_n$$

$$L(Q, f, \alpha) - L(P, f, \alpha) = [\alpha(x^*) - \alpha(x_{i-1})] \cdot \inf_{[x_{i-1}, x^*]} f(x) = m_i^-$$



$$+ \ [\alpha(x_i) - \alpha(x^*)] \cdot \inf_{[x^*, x_i]} f(x) = m_i^+$$

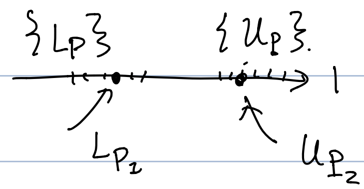
$$- \ [\alpha(x_i) - \alpha(x_{i-1})] \inf_{[x_{i-1}, x_i]} f(x) \ \text{“} \ m_i$$

$$m_i^+ = \inf_{[x^*, x_i]} f(x) \geq \inf_{[x_{i-1}, x_i]} f(x) = m_i$$

$$m_i^- \geq m_i$$

$$= [\alpha(x^*) - \alpha(x_{i-1})](m_i^- - m_i)$$
$$+ [\alpha(x_i) - \alpha(x^*)](m_i^+ - m_i)$$
$$\geq 0.$$

#.



<u>Thm (6.5. in Rudin)</u> $\quad L(f,\alpha) \leq U(f,\alpha)$

<u>Pf</u>: Suffice to prove that $\forall P_1, P_2$ partitions.

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha).$$

Let $P_1 \cup P_2$ be the common refinement. Then.

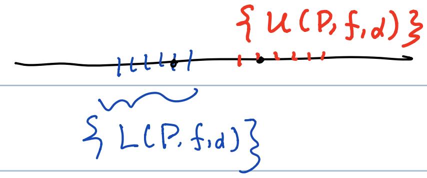
$$L_{P_1} \leq L_{P_1 \cup P_2} \leq U_{P_1 \cup P_2} \leq U_{P_2}$$

Hence $\quad L(P_1, f, \alpha) \leq \inf_{P_2} U(P_2, f, \alpha). = U(f, \alpha).$

then take sup over P_1, we gets

$$L(f,\alpha) \leq U(f,\alpha).$$

#.



Thm (Cauchy Condition)

$f \in R(\alpha) \iff \forall \varepsilon > 0, \ \exists P$ partition, such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon.$$

Pf: $\Leftarrow$ ✓ $\because$

$$L(P,f,\alpha) \underset{\text{def of sup}}{\leq} L(f,\alpha) \underset{\text{Thm 6.5}}{\leq} U(f,\alpha) \underset{\text{def of inf}}{\leq} U(P,f,\alpha), \quad \forall P \text{ partition.}$$

$\therefore \quad 0 \leq U(f,\alpha) - L(f,\alpha) \leq U(P,f,\alpha) - L(P,f,\alpha).$

- since $\forall \varepsilon > 0$, $\exists P$ partition, s.t. $U_P - L_P < \varepsilon$.

$\therefore \quad 0 \leq U(f,\alpha) - L(f,\alpha) < \varepsilon \qquad \forall \varepsilon > 0.$

$\therefore \quad U(f,\alpha) = L(f,\alpha). \qquad \Rightarrow f \in R(\alpha).$

$\Rightarrow$ Let $\int f\, d\alpha$ denote the common value $U(f,\alpha)$
$= L(f,\alpha).$

$\because \quad \int f\, d\alpha = \inf \{ U(P,f,\alpha) \mid P \text{ partition} \}$

$\therefore \quad \exists P_1$, s.t.

$$U(P_1,f,\alpha) - \int f\, d\alpha < \frac{\varepsilon}{2}.$$

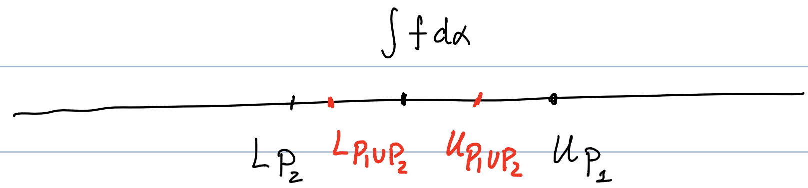
$\because \quad \int f\, d\alpha = \sup \{ L(P,f,\alpha) \mid P \text{ partition} \}.$

$\therefore \quad \exists P_2$, s.t.

$$\int f\, d\alpha - L(P_2,f,\alpha) < \frac{\varepsilon}{2}.$$

$\therefore$ Let $P = P_1 \cup P_2$, then.

$\int f\, d\alpha - L(P,f,\alpha) < \frac{\varepsilon}{2}, \quad U(P,f,\alpha) - \int f\, d\alpha < \frac{\varepsilon}{2}. \quad \Rightarrow \cdots$ #.

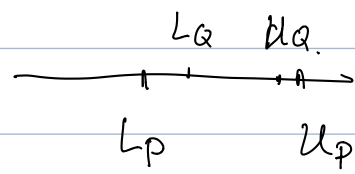


$$U_P = U(P, f, \alpha). \qquad L_P = L(P, f, \alpha).$$

<u>Lemma</u> (Thm 6.7 Rudin).

(a). if $U_P - L_P < \varepsilon$, then for any Q, refinement of P, we have

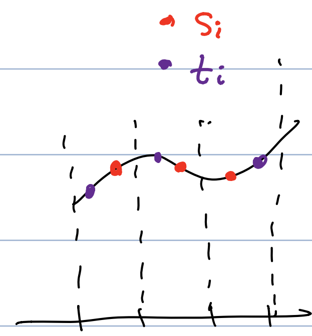
$$U_Q - L_Q < \varepsilon.$$



(b). If $U_P - L_P < \varepsilon$, and.

let $s_i, t_i \in [x_{i-1}, x_i]$, $\forall\, i = 1, \cdots, n$.

Then
$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \cdot \Delta\alpha_i < \varepsilon.$$



Pf of (b): $\because |f(s_i) - f(t_i)| \leqslant M_i - m_i$

$$\therefore \sum |f(s_i) - f(t_i)| \cdot \Delta\alpha_i \leq \sum (M_i - m_i) \cdot \Delta\alpha_i$$
$$= U_P - L_P < \varepsilon.$$

(c) If $f \in R(\alpha)$, and $U_P - L_P < \varepsilon$, $s_i \in [x_{i-1}, x_i]$ $\forall i$

Then
$$\left| \sum f(s_i) \cdot \Delta\alpha_i - \int f\,d\alpha \right| < \varepsilon.$$

<u>Rmk</u>: For a more detail comparison between Riemann sum using sample points $\sum f(s_i)\,\Delta\alpha_i$, and Darboux sum $L(P, f, \alpha)$, $U(P, f, \alpha)$. See Ross (for classical Riemann integral, no α).

Thm: If f is continuous on $[a,b]$, then $f \in R(\alpha)$ on $[a,b]$.

Pf: Let $\varepsilon > 0$ be given. Since f is continuous on a compact set $[a,b]$, hence f is uniformly continuous. Hence, $\forall \eta > 0$, $\exists\ \delta(\eta) > 0$. s.t. $\forall\ |x-y| < \delta(\eta)$, then $|f(x) - f(y)| < \eta$.

Consider a partition P, such that $\Delta x_i < \delta(\eta)$.

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta\alpha_i.$$

for some p_i, q_i

$$0 \le M_i - m_i = \sup_{[x_{i-1}, x_i]} f(x) - \inf_{[x_{i-1}, x_i]} f(x) = f(p_i) - f(q_i) \le \eta.$$

$$\le \sum_{i=1}^{n} \eta \cdot \Delta\alpha_i = \eta \cdot \sum_{i=1}^{n} \Delta\alpha_i = \eta\,(\alpha(b) - \alpha(a)).$$

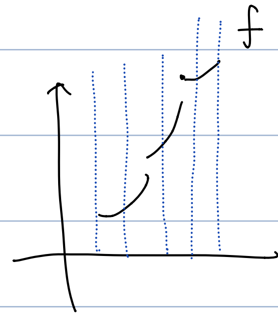
Now choose η. such that $\eta\,(\alpha(b) - \alpha(a)) \le \varepsilon$, this finishes the proof. #

Thm: If f is monotonic on $[a,b]$, and α is continuous (and also increasing), then $f \in R(\alpha)$.

Pf: Let $\varepsilon > 0$. Consider partition P_n with n segments ($\forall n > 0$ integer) such that $\Delta\alpha_i = \dfrac{\alpha(b) - \alpha(a)}{n}$. This is achievable using continuity of α and intermediatrate value thm.

Then.

$$\mathcal{U}(P_n, f, \alpha) - L(P_n, f, \alpha).$$



$$= \sum_{i=1}^{n} (M_i - m_i) \cdot \Delta\alpha_i$$

$$= \sum_{i=1}^{n} \left[f(x_i) - f(x_{i-1}) \right] \cdot \frac{\alpha(b) - \alpha(a)}{n}.$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \cdot \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1}) \right)$$

$$= \frac{\alpha(b) - \alpha(a)}{n} \cdot \left(f(b) - f(a) \right).$$

We can take n large enough, so that the differeme $\mathcal{U}_{P_n} - L_{P_n} < \varepsilon$. #.