Last time:

$f^{(n)}(x)$ exist $\forall x \in \mathbb{R}$, $\forall n \in \mathbb{N}$.

- Assume $f: \mathbb{R} \to \mathbb{R}$ is a smooth function. (C^∞)

Then for any $N \geq 0$ integer, $\forall x_0 \in \mathbb{R}$, we have

N-th order Taylor expansion:

$$P_{x_0, N}(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.$$

the unique degree $(\leq N)$ polynomial, such that

$$P^{(k)}(x_0) = f^{(k)}(x_0) \qquad \forall k = 0, 1, 2, \cdots, N.$$

- Taylor Theorem:

here, the derivative is evaluated at ξ.

$$f(x) - P_{x_0, N}(x) = \underbrace{\frac{f^{(N+1)}(\xi)}{(N+1)!} (x-x_0)^{N+1}}_{\text{Remainder term.}}$$

ξ is a point between x and x_0.

$\left(\begin{cases} x < \xi < x_0, \text{ or} \\ x_0 < \xi < x \end{cases} \right)$

Today: Taylor Series $(N \to \infty)$. of f at x_0.

$$P_{x_0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n.$$

warning: ① given an input $x \in \mathbb{R}$, the series may not be convergent

② even if the series is convergent for $x \in \mathbb{R}$, it may not equal to $f(x)$.

$\boxed{\text{Power series}}$: series of the form: $\sum_{n=0}^{\infty} C_n \cdot (x-x_0)^n$

- Radius of convergence R:

$R = \sup \{ r \geq 0,$ s.t. if $|x - x_0| \leq r$, the series converges. }.

it may ~~app~~ happen that the series is convergent only for $x = x_0$, then $R = 0$.

or ~~at~~ the series converges for all x, then $R = \infty$.

section "power series"

· <u>Theorem (3.xx. Rudin)</u>

Let $\alpha = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$. Let $R = \frac{1}{\alpha}$.

Then. · if $|x - x_0| < R$, the series converges.

· if $|x - x_0| > R$, the series diverges.

Pf: Use root test for convergences.

If $|x - x_0| < R$, then for $\sum c_n \cdot (x - x_0)^n$,

$$\limsup |c_n (x - x_0)^n|^{\frac{1}{n}} = \limsup_{n\to\infty} \left(|c_n|^{\frac{1}{n}} \cdot |x - x_0| \right)$$

$$= |x - x_0| \cdot \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \frac{|x - x_0|}{R} < 1.$$

∴ convergent by root test.

The divergent case is exercise. #

Ex: (1). $f(x) = \frac{1}{1-x}$. Taylor expansion at $x = 0$.

instead of computing $f^{(n)}(0)$, we use

· $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \cdots\cdots$ $\forall$ $|x| < 1$.

(if $x = 1$, $LHS = \frac{1}{0}$, $RHS = 1+1+1+\cdots$, neither side is a real number.

if $x = -1$, $LHS = \frac{1}{2}$, $RHS = 1-1+1-1 \cdots\cdots$ not convergent.)

· Since RHS is valid in a neighborhood of $x=0$,

$\therefore$ this is the Taylor expansion around $x=0$.

- Taylor expand around $x_0 = -1$, we can define.

$u = x - x_0 = x+1$, then. $x = u-1$.

$$\frac{1}{1-x} = \frac{1}{1-(u-1)} = \frac{1}{2-u} = \frac{1}{2}\,\frac{1}{1-\frac{u}{2}}$$

$$= \frac{1}{2}\cdot\left(1+\frac{u}{2}+\left(\frac{u}{2}\right)^2+\left(\frac{u}{2}\right)^3+\cdots\right)$$

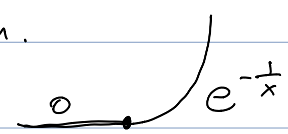
for $\left|\frac{u}{2}\right|<1$

$$= \frac{1}{2} + \frac{1}{2^2}(x-x_0) + \frac{1}{2^3}(x-x_0)^2 + \cdots + \frac{1}{2^{n+1}}(x-x_0)^n + \cdots$$

$x_0 = -1$.

<u>Ex 3</u>: · Let $\varphi(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x}} & x > 0. \end{cases}$ smooth.

$\varphi^{(n)}(0) = 0$. $\forall n = 0, 1, 2, \cdots$



· Taylor series of φ at $x=0$. is

$$P_{x_0}(x) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(0)}{n!}(x)^n = 0. \neq \varphi(x).$$

· In general, if $f(x) = \sum_{n=0}^{\infty} C_n (x-x_0)^n$ for $|x-x_0|<R$.

then $f(x) + \varphi(x-x_0)$. would have the same Taylor series at x_0.

<u>Def</u>: A smooth function $f: (a,b) \to \mathbb{R}$ is <u>real analytic</u>, if $\forall x_0 \in (a,b)$, $f(x) = \sum_n C_n \cdot (x-x_0)^n$

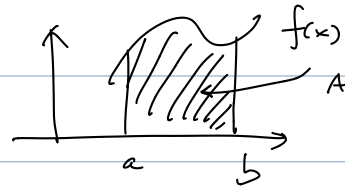
for x in some neighborhood of x_0.

Prop: In fact, near x_0, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$.

- Riemann - Stieltjes integral. (another useful notion Lebesgue integral.)

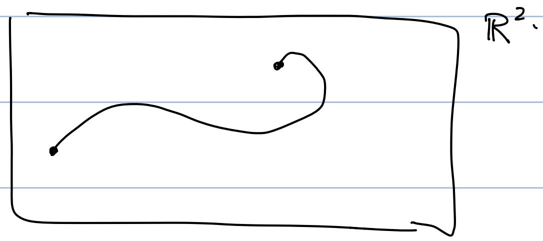
- Motivation:
  - compute the area of certain irregular shape. e.g. area of a disk.

    area underneath a curve



Area = $\int_a^b f(x)\,dx$.

  - compute the length of a curve.



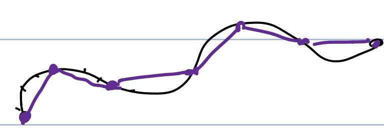
- ....

Method: use approximation by nice regular objects, and then take limit.

Example (when approximation fails, i.e. limit does not exist):

(1) length of a curve.

- sample points on the curve and approximate the curve by



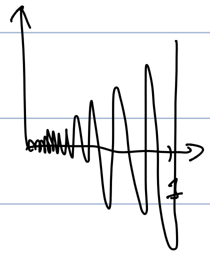
segments.

- Hope the total length of the segments converge as "more sampling points" are take.

- Fractal curve. (Mandelbrot set)
- space filling curve of a square.

- ex. the length of graph $f(x) = \begin{cases} 0 & x = 0 \\ x \cdot \sin\left(\frac{1}{x^2}\right) & x > 0. \end{cases}$

$x \in [0, 1]$



continuous

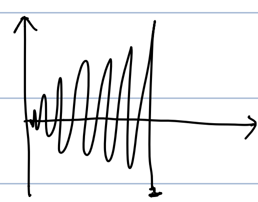
length of the graph $\int_0^1 \sqrt{1 + f'(x)^2}\, dx.$

$x > 0.$ $f'(x) = \sin\left(\frac{1}{x^2}\right) + x \cdot \left(-\frac{1}{x^3}\right) \cos\left(\frac{1}{x^2}\right)$

$= \sin\left(\frac{1}{x^2}\right) - \frac{1}{x^2} \cdot \cos\left(\frac{1}{x^2}\right).$

near $x = 0$, $\int_0^1 \frac{1}{x^2} \left|\cos\left(\frac{1}{x^2}\right)\right| dx.$ not convergent.

(2). $\underline{\int_0^1 x \cdot \sin\left(\frac{1}{x^2}\right) dx. = ?}$ does it make sense ?



Ans: yes.
$\int_a^b f\, dx$ exists, if f is continuous on $[a,b]$.

$\int_0^1 x\left(2 + \sin\left(\frac{1}{x^2}\right)\right) dx < \int_0^1 x \cdot 3\, dx$

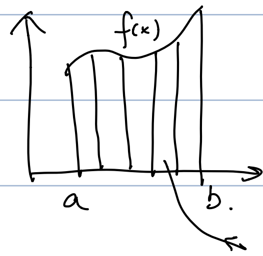
V

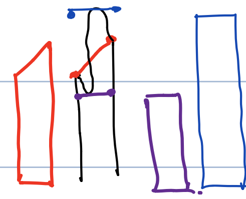
$\int_0^1 x \cdot 1 \quad dx$

$S \subset [0,1]^2$

$\text{Area}(S) = ?$

- Roughly speaking:
  - cut the area under the graph of f into thin stripes.





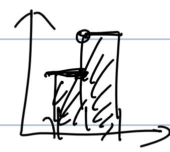
  - approximate each strip by something regular
  - then take limit as we use more & more stripes.

- Def: A paritition P of interval $[a,b]$, is

  $a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n = b$.

  We define $\Delta x_i = x_i - x_{i-1}$, $i = 1, \cdots, n$.

- For Riemann-integral, we consider any $f : [a,b] \to \mathbb{R}$ that is bounded (not necessarily continuous).



- Given a bounded real valued function $f : [a,b] \to \mathbb{R}$, given a partition $P = (a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b)$,

we define

- $$U(P, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$$

  $M_i = \sup\{ f(x) \mid x \in [x_{i-1}, x_i] \}$

  $\Delta x_i = x_i - x_{i-1}$.

- $L(P,f) = \sum_{i=1}^{n} m_i \, \Delta x_i$

$m_i = \inf \{ f(x) \mid x \in [x_{i-1}, x_i] \}$.

$\mathcal{P}$ curly font Capital P.

- $\mathcal{U}(f) := \inf_{\mathcal{P}: \text{partition of } [a,b]} \mathcal{U}(P,f)$. also denoted as $\overline{\int_a^b} f \, dx$

$L(f) := \sup_{\mathcal{P} \ldots} L(P,f) = \underline{\int_a^b} f(x) \, dx$.

We say f is integrable, if $\mathcal{U}(f) = L(f)$.

Simple facts: if $m \le f(x) \le M$, $\forall x \in [a,b]$, then $\forall \mathcal{P}$ partition,

$$m \cdot (b-a) \le L(P,f) \le \mathcal{U}(P,f) \le M(b-a).$$

- A generalization:

Rie... $\int f(x) \, dx$

Stieltje replaces dx by something more general.

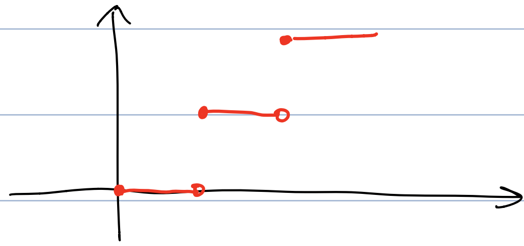
- Let $\alpha : [a,b] \to \mathbb{R}$ be a monotone increasing function.

- Given a partition $P = \{ a = x_0 \le x_1 \le \cdots \le x_n = b \}$ define $\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \ge 0$.

- Repeat the previous definitions, $\Delta x_i \rightsquigarrow \Delta \alpha_i$.
  we get $U(P, f, \alpha)$, $U(f, \alpha)$. ---
  If $U(P, \alpha) = L(P, \alpha)$, we say f is Riemann - Stieltjes integrable w.r.t. α. Integral is $\int f(x)\, d\alpha(x)$,
  and write. $f \in \mathcal{R}(\alpha)$.

<u>Ex</u>: (1) $\alpha(x) = x$, this reduces to Riemann integral.

(2) $\alpha(x) = \lfloor x \rfloor$



then. if f is continuous. then.

$$\int f(x)\, d\alpha(x) = \sum_{n \in \mathbb{Z}} f(n).$$

we can talk about summation & integral in the same framework.