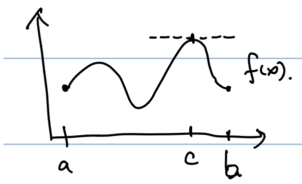
① Review : 1. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

2. MVT : (a) (Rolle) $f(a) = f(b) \Rightarrow \exists\, c \in (a,b)$, s.t. $f'(c) = 0$.

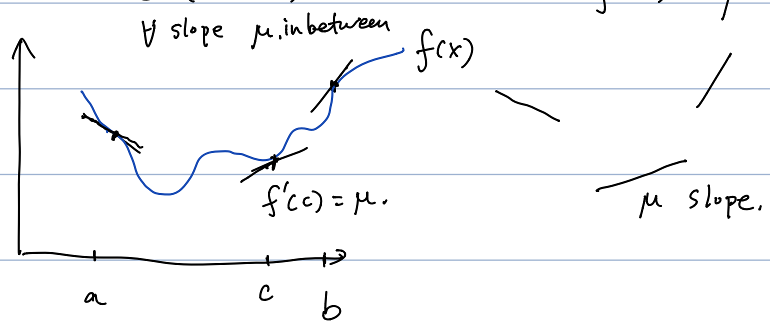


(b) (General) $\exists\, c \in (a,b)$. s.t.

$$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c).$$

(c). (common) $\exists\, c \in (a,b)$. $f(b) - f(a) = (b-a) \cdot f'(c)$.

3. IVT : if $f'(a) < f'(b)$, then $\forall \mu$, $f'(a) < \mu < f'(b)$, we have $c \in (a,b)$, s.t. $f'(c) = \mu$.



4. L'Hopital Rule.

Today : Higher Derivatives. Taylor Theorem. Examples.

Taylor Series / Power Series. ← in Rudin ch 3.

Higher Derivatives : If $f'(x)$ is differentiable at X_0, then we define $f''(x_0) = (f')'(X_0)$. Similarly, if the (n-1)-th derivative $f^{(n-1)}$ exists and is differentiable at X_0, we define $f^{(n)}(X_0) = \left(f^{(n-1)}\right)'(X_0)$.

Ex : : $f(x) = x^2$. $f'(x) = 2X$, $f''(x) = 2$.

- in general. $(x^n)^{(k)} = n(n-1)\cdots(n-k+1)\cdot x^{n-k} = \frac{n!}{(n-k)!}x^{n-k}$.

  notation for k-th derivative.

- if $f(x) = e^x$, $f'(x) = e^x \Rightarrow f^{(n)}(x) = e^x$

- if $f(x) = \sin x = \frac{e^{ix} - e^{-ix}}{2i}$,

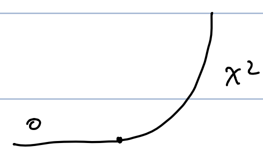
$$f^{(n)}(x) = \begin{cases} \cos x & n = 4k+1 \\ -\sin x & n = 4k+2 \\ -\cos x & n = 4k+3 \\ \sin x. & n = 4k \end{cases} \qquad k \in \mathbb{N}.$$

- $(\log x)' = \frac{1}{x}$,

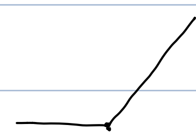
$\underline{n < 0}$ $\quad (x^n)^k = n(n-1)\cdots(n-k+1)\cdot x^{n-k}$.

$\underline{\text{Ex}}$: (higher derivative exists, but not to all order)

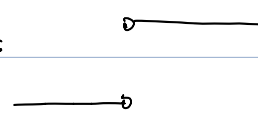
$$f(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & x > 0. \end{cases}$$

f: 

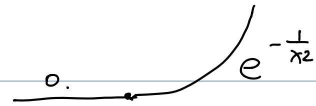
$$f'(x) = \begin{cases} 0 & x \leq 0 \\ 2x & x > 0. \end{cases}$$

f': 

$$f''(x) = \begin{cases} 0 & x < 0 \\ \text{not defined.} & x = 0 \\ 2. & x > 0 \end{cases}$$

f'': 

Ex: $f(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases}$



$e^{-\frac{1}{x^2}} \to 0$ as $x \to 0^+$ at a ~~rate~~ faster rate than x^k for any k.

So, $f^{(k)}(x) \to 0$ as $x \to 0^+$.

$\Rightarrow f^{(k)}(0) = 0. \quad \forall k \in \{1, 2, \cdots\}$.

and $f^{(k)}(x)$ is continuous $\forall x \in \mathbb{R}$.

Def: $f(x)$ is a smooth function on (a,b) if $\forall x \in (a,b)$, $\forall k \in \{1, 2, \cdots\}$.
$f^{(k)}(x)$ exists.

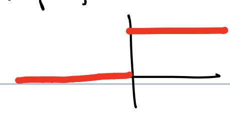
$\forall x > 0$

$$f'(x) = \left(-\frac{1}{x^2}\right)' \cdot e^{-\frac{1}{x^2}} = -(-2)\, x^{-2-1} \cdot e^{-\frac{1}{x^2}} = x^{-3} \cdot e^{-\frac{1}{x^2}}$$

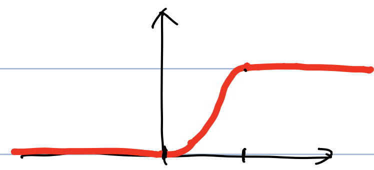
use Lemma: $\lim\limits_{u \to \infty} e^{-u} \cdot u^k = 0. \quad \forall k \in (0, \infty)$.

as exercise.

usual step function

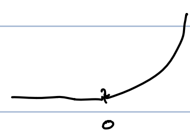


(2). "smooth step function"

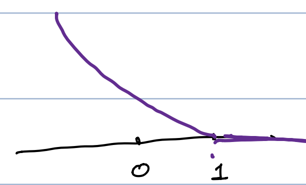


$$g(x) = \frac{f(x)}{f(x) + f(1-x)}$$

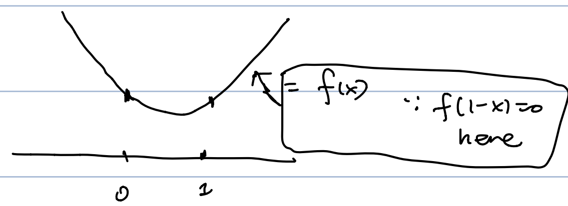
$f(x)$:



, $f(1-x)$:



$f(x) + f(1-x)$:



$$g(x) = \begin{cases} 0 & x \le 0 \quad \leftarrow \because \text{numerator } f(x) = 0 \\ \text{between } (0,1). & x \in (0,1) \\ 1 & x \ge 1 \end{cases}$$

$\nwarrow \because f(1-x) = 0$ in the denominator.

$g(x)$ is smooth $\because$ $f(x)$ is smooth and denominator $f(x)+f(1-x)$ is strictly positive & smooth.

(3) " Smooth bump function "

$$g(x) = \frac{f(x)\,f(1-x)}{f(x)+f(1-x)} = \begin{cases} =0 & x \leq 0 \\ >0 & x \in (0,1) \\ =0 & x \geq 1 \end{cases}$$

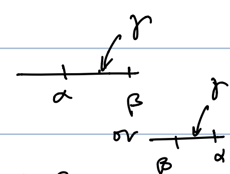
- **Taylor Theorem:** Let $f : [a,b] \to \mathbb{R}$, assume

  $f^{(n-1)}(x)$ exists and is continuous $\forall\ x \in [a,b]$.

  $f^{(n)}(x)$ exists $\forall\ x \in (a,b)$.

Then $\forall\ \alpha \in [a,b]$, define

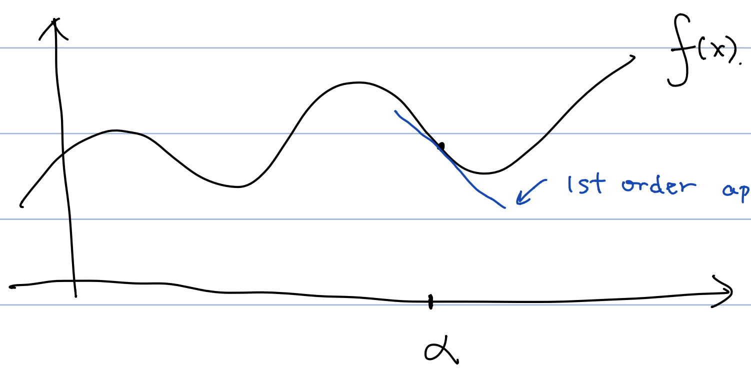
$$P_\alpha(x) = \sum_{k=0}^{n-1} \cdot \frac{f^{(k)}(\alpha)}{k!} \cdot (x-\alpha)^k.$$



Then $\forall\ \beta \in [a,b]$, if $\beta \neq \alpha$, $\exists\ \gamma$ between α and β, s.t.

$$f(\beta) - P_\alpha(\beta) = \frac{f^{(n)}(\gamma)}{n!}(\beta-\alpha)^n.$$

Intuition:



want to describe $f(x)$ near α, with some approximation.

0-th order level: value $f(\alpha)$.

1st order : $f(\alpha) + f'(\alpha)\cdot(x-\alpha)$.

2nd order : $P_{\alpha,2}(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(\alpha)}{2!}(x-\alpha)^2$

$P_{\alpha,2}(\alpha) = f(\alpha)$, $P'_{\alpha,2}(\alpha) = f'(\alpha)$, $P''_{\alpha,2}(\alpha) = f''(\alpha)$.

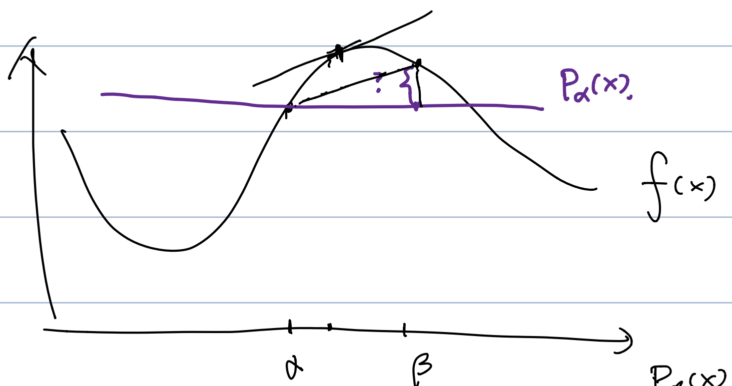
Taylor theorem is about the "error term",

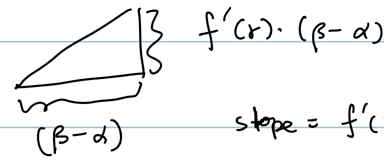
$$f(x) - P_{\alpha, n-1}(x).$$

· In the case $n=1$, then $P_{\alpha,n-1}(x) = f(\alpha)$. constant fcn.

$$f(\beta) - P_{\alpha}(\beta) = f(\beta) - f(\alpha) = (\beta - \alpha)\cdot f'(\gamma)$$

by MVT, $\exists\, \gamma \in (\alpha, \beta)$ s.t.





Write $P(x) = P_{\alpha,n-1}(x)$ (" $P_\alpha(x)$ ") for short. Assume $\alpha < \beta$.

Pf: Let M be the number defined by

$$f(\beta) - P(\beta) = (\beta - \alpha)^n \cdot M.$$

· Define function

(*) $$g(x) = f(x) - P(x) - M\cdot(x-\alpha)^n.$$

then $g(\beta) = f(\beta) - P(\beta) - M\cdot(\beta-\alpha)^n = 0$ by the choice of M.

and $g(\alpha) = f(\alpha) - P(\alpha) - 0 = 0$.

$\because$ $P(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \cdots + \dfrac{f^{(n-1)}(\alpha)}{(n-1)!}(x-\alpha)^{n-1}$

$\therefore$ $P(\alpha) = f(\alpha)$

· We want to show that $M = \dfrac{f^{(n)}(\gamma)}{n!}$ for some $\gamma \in (\alpha, \beta)$.

By (*),

$$g^{(n)}(x) = f^{(n)}(x) - n!\, M.$$

$\because$ $P(x)$ is degree $n-1$ polynomial in x, $\therefore$ $P^{(n)}(x) = 0$.

$\forall\, x \in (a,b)$. Then, if we can show, $\exists\, \gamma \in (\alpha,\beta)$ s.t.

$g^{(n)}(\gamma) = 0$, then we are done.

- Also note : $g(\alpha) = 0$, and. $\forall\ k \in \{1, \cdots, n-1\}$

$$g^{(k)}(x) = f^{(k)}(x) - P^{(k)}(x) - M \cdot \left[(x-\alpha)^n\right]^{(k)}.$$

$$\Rightarrow g^{(k)}(\alpha) = f^{(k)}(\alpha) - P^{(k)}(\alpha) - 0 = 0.$$

- Now, we are ready to apply MVT.

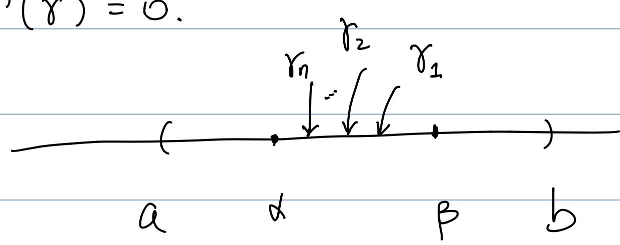
(1) $g(\alpha) = 0$, $g(\beta) = 0$ $\Rightarrow \exists\, \gamma_1 \in (\alpha, \beta)$, s.t. $g'(\gamma_1) = 0$.

(2) $g'(\alpha) = 0$, $g'(\gamma_1) = 0$, $g''(x)$ exists in (α, γ_1) $\Rightarrow \exists\, \gamma_2 \in (\alpha, \gamma_1)$, s.t. $g''(\gamma_2) = 0$.

$\vdots$

(n): $g^{(n-1)}(\alpha) = 0$, $g^{(n-1)}(\gamma_{n-1}) = 0$, $g^{(n)}(x)$ exists $\forall\, x \in (\alpha, \gamma_{n-1})$ $\overset{\text{MVT}}{\Rightarrow} \exists\, \gamma_n \in (\alpha, \gamma_{n-1})$, s.t. $g^{(n)}(\gamma_n) = 0$

Let $\gamma = \gamma_n$, then $g^{(n)}(\gamma) = 0$. #.



- Taylor Series for a smooth function.

If f is a smooth function on (a, b)., and. $\alpha \in (a, b)$. We can form the <u>Taylor series</u> :

$$P_\alpha(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k.$$

<u>Warning</u> : ① the right hand side is not guaranteed to converge.

② Even if it converges., it may not equal to $f(x)$.

Next time: power series and radius of convergence.