Math 104 12:30 – 2:00

midterm 1: next Thu. covers everything till end of this week.
practice problem given before weekends.

Last time: subsequence.

Thm: (s_n) a seq. $t \in \mathbb{R}$ is the limit of a subseq in (s_n) iff $\forall \varepsilon$, there is infinitely many terms in (s_n) inside $(t-\varepsilon, t+\varepsilon)$.

sketch of proof: Take $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{k}, \cdots$. $\varepsilon_k = \frac{1}{k}$.

Define subseq. $A_k = \{ n \mid |s_n - t| < \varepsilon_k \}$.

$A_1 \supseteq A_2 \supseteq A_3 \cdots$

$A_k = \{ n_{k1} < n_{k2} < n_{k3} < \cdots \}.$

Consider the array

$A_1: \quad S_{n_{11}}, \quad S_{n_{12}}, \quad S_{n_{13}}, \cdots\cdots$

$A_2: \quad S_{n_{21}}, \quad S_{n_{22}}, \quad S_{n_{23}}, \cdots\cdots$

$A_3 \quad S_{n_{31}}, \quad S_{n_{32}}, \quad S_{n_{33}}, \cdots\cdots$

indices: n_{ki}

1, 2, 3, 4, ...

2, 3, 4, 6, ...

4, 8, ...

$\boxed{\text{Cartan "diagonal" trick}}$

- $n_{11} < n_{12} < \cdots\cdots$ horizontally, indices are increasing

$n_{1i}\ k\cdots \le n_{ki} \le n_{k+1,i} \le n_{k+2,i} < \cdots$

$\Rightarrow \quad n_{11} < n_{22} < n_{33} < \cdots,$ $\quad (S_{n_{kk}})_k$ is the desired subseq #.

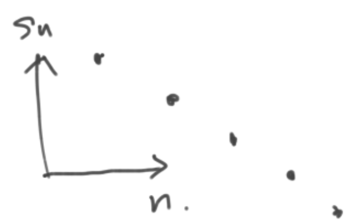
Thm 11.4 Every seq (s_n) has a monotone subseq.

Pf: We call a term s_n in the seq. **dominant**,

if $\forall m > n$, $s_n > s_m$.

case 1: there are infinitely many dominant terms.

$s_{n_1}, \; s_{n_2}, \; s_{n_3}, \cdots$

Ex: 

a (strictly) decreasing seq, then each s_n is a dominant term.

$s_{n_1} > s_{n_2} > s_{n_3} > \cdots$

The dominant terms forms a monotone decreasing seq.

<u>case 2</u> : there are only finitely many dominant terms.

$S_{n_1}, \cdots, S_{n_k}$.

Then we have an n_0, s.t. $\forall n > n_0$, ~~the~~ S_n is not a dominant term.

S_n is not dominant $\Leftrightarrow$ $\exists\, m > n$, s.t. $S_m \geq S_n$.

Hence, we can construct a monotone increasing sequence, by induction.

- Pick n_1 to be any integer $> n_0$.
- suppose $n_1 < n_2 < \cdots < n_k$ are constructed s.t. $S_{n_1} \leq S_{n_2} \leq \cdots \leq S_{n_k}$, then we pick $n_{k+1} > n_k$, s.t. $S_{n_{k+1}} \geq S_{n_k}$. Such n_{k+1} exists, since S_{n_k} is not dominant.

Thm: Every bounded seq has a convergent sub seq.

Pf: use Thm 11.4, pick a monotone subseq. Since this subseq is also bounded, hence its convergent. #.

Pf #2: Assume $S_n \in [-M, M]$, $\forall n \in \mathbb{N}$.

Let $I_0 = [-M, M]$, Assume that we have closed interval

- $I_0 \supset I_1 \supset I_2 \cdots \supset I_k$, s.t. $\frac{1}{2}|I_{k-1}| = |I_k|$
- and each I_i, $0 \le i \le k$, contain infinitely many points in (S_n).

$I_k = [a_k, b_k]$.

$\Rightarrow$ we construct I_{k+1}. as one half of I_k,

$I_k^- = [a_{k+1}, \frac{1}{2}|I_k| + a_k]$, $[\frac{1}{2}|I_k| + a_k, b_k] = I_k^+$.

At least one of the intervals I_k^+, I_k^- contain ∞ many terms in (S_n), take I_{k+1} be such an interval. Then, use again the diagonal arguments:

$$A_1 = \{n \mid S_n \in I_1\}$$
$$A_2 = \{n \mid S_n \in I_2\},$$

$(S_{n_k})_k$ ''

$A_1 \supset A_2 \supset A_3 \supset \cdots\cdots$. Hence, we get a subseq $(S_{n_{kk}})_k$. such that $S_{n_k} \in I_k$. Since $\forall\, l > k$, $S_{n_l} \subset I_l \subset I_k$, we have $|S_{n_k} - S_{n_l}| < |I_k| = 2^{-k}|I_0|$.

Hence $(S_{n_k})_k$ is a Cauchy seq.

$\Rightarrow (S_{n_k})$ is convergent.

Def: Let (S_n) be a seq, $t \in \mathbb{R} \cup \{+\infty, -\infty\}$ is said to be a subseq limit of S_n, if $\exists$ a subseq of S_n whose limit is t.

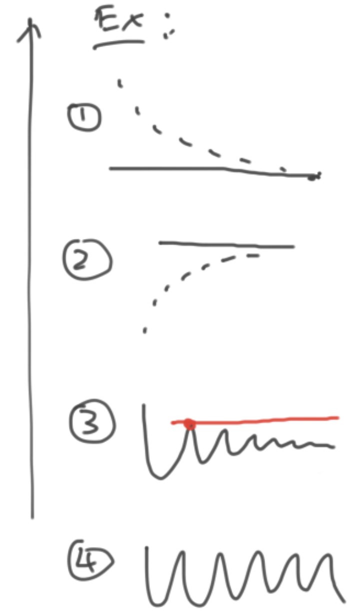
[Recall, $\lim_n t_n = +\infty$ means that, $\forall M > 0$, $\exists N > 0$ s.t. $\forall n > N$, $t_n > M$.

(in a sense, $\frac{1}{M}$ is the "distance" to $+\infty$)].

Thm 11.7: Let (S_n) be any seq. then $\limsup S_n$ and $\liminf S_n$ are subseq limits.

Pf: assume (S_n) is bounded, hence $\limsup S_n$ exists in $\mathbb{R}$.

- Let $\alpha = \limsup S_n$. We only need to show, $\forall \varepsilon > 0$, $(\alpha - \varepsilon, \alpha + \varepsilon)$ contains infinitely many terms in (S_n).



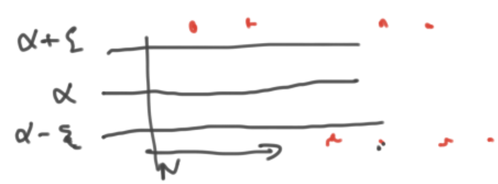
① Let $A_n = \sup\limits_{m \geq n} (S_n)$ be the seq of the sup of the tail. Then $\lim A_n = \alpha$, $A_n \searrow$.

Thus, $\exists N > 0$, s.t. $\forall n > N$,

$$|A_n - \alpha| < \frac{\varepsilon}{2}.$$

i.e. $\alpha - \frac{\varepsilon}{2} < \underline{A_n} < \alpha + \frac{\varepsilon}{2}$ $\quad \underline{\forall n > N}$.

② Assume, that $(\alpha - \varepsilon, \alpha + \varepsilon)$ only contain finitely many terms in (S_n), in particular, $\exists N_0$, s.t. $\forall n > N_0$, $S_n \notin (\alpha - \varepsilon, \alpha + \varepsilon)$



[Exercise: figure out the contradiction between ①, ②]

- $S = \{ t \in \mathbb{R} \cup \{+\infty, -\infty\} \mid t$ is a sub seq limit of $(s_n) \}$.

Thm: ① S is not empty. ($\because$ $\limsup s_n$ and $\liminf s_n$ are inside S. note. they may coincide. they may be $+\infty$ or $-\infty$)

② $\sup(S) = \limsup s_n$

$\inf(S) = \liminf s_n$.

③ ✓ S has a single element $\Leftrightarrow$ $\lim\limits_{n\to\infty} s_n$ exists (in $\mathbb{R} \cup \{+\infty, -\infty\}$).

Pf ②:
- we know $\limsup s_n \in S$, by previous thm.
- only need to show, $\forall t \in S$, $t \leq \limsup s_n$.

Let $(s_{n_k})_k$ be ~~the~~ a sub seq. that converges to t.

$\because$ (s_{n_k}) is a subseq of (s_n), hence.

$\therefore$ $\limsup s_{n_k} \leq \limsup s_n$.

$\because$ (s_{n_k}) converges. $\therefore$ $\limsup s_{n_k} = \lim s_{n_k} = t$. $\Rightarrow$ $t \leq \limsup s_n$. #

• Notion : "closed subset".

A subset $S \subset \mathbb{R}$ is a closed subset, if $\forall$ convergent sequence in S, the limit also belong to S.

i.e. if (s_n) is a seq. s.t. ① $s_n \in S \ \forall n$ ② $\lim s_n = s \in \mathbb{R}$

then $\lim s_n \in S$.

Ex: $[0,1]$ is closed.

$(0,1)$ is not closed. e.g. $s_n = \frac{1}{n}$.

Thm (11.18): Let (s_n) be any bounded seq. Let S be the set of subseq. limit. Then S is closed.