1. (20 points, 4 points each) Topology on $\mathbb{Q}$. You may use the fact that the topology on $\mathbb{Q}$ constructed from the induced metric on $\mathbb{Q}$ as a subset of $\mathbb{R}$ is equivalent to the induced topology on $\mathbb{Q}$ as a subset of $\mathbb{R}$.

   Answer the following question and **justify your answer.**

   (1) Is the set $(-1, 1) \cap \mathbb{Q}$ open in $\mathbb{Q}$? Is it closed in $\mathbb{Q}$? open, not closed.

   (2) Is the set $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ open in $\mathbb{Q}$? Is it closed in $\mathbb{Q}$? open and closed.

   (3) Is the set $(0, 2) \cap \mathbb{Q}$ connected? $(0,2) \cap \mathbb{Q} = [(0,\alpha) \cap \mathbb{Q}] \cup [(\alpha, 2) \cup \mathbb{Q}]$, $\forall \alpha \in (0,2)$ irrational.

   (4) Find a subset $K \subset \mathbb{Q}$, such that K is closed and bounded in $\mathbb{Q}$, but not compact. $K = [0,1] \cap \mathbb{Q}$

   (5) Find a subset $K \subset \mathbb{Q}$, such that K is compact and K is an infinite set. $K = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$

2. (20 points, 2 each) True or False. **No justification is needed.**

   Let $f : X \to Y$ be a continuous map between metric spaces. Let $A \subset X$ and $B \subset Y$.

   F (1) If A is open, then $f(A)$ is open.

   F (2) If A is closed, then $f(A)$ is closed.

   F (3) If A is bounded, then $f(A)$ is bounded.

   T (4) If A is connected, then $f(A)$ is connected.

   T (5) If A is compact, then $f(A)$ is compact.

   T (6) If B is open, then $f^{-1}(B)$ is open.

   T (7) If B is closed, then $f^{-1}(B)$ is closed.

   F (8) If B is bounded, then $f^{-1}(B)$ is bounded.

   F (9) If B is connected, then $f^{-1}(B)$ is connected.

   F (10) If B is compact, then $f^{-1}(B)$ is compact.

3. (15 points) Prove that the finite union of compact subsets is compact. More precisely, let X be a metric space, and let $K_1, \cdots, K_n$ be any compact subsets of X. Show that $K_1 \cup \cdots \cup K_n$ is a compact subset of X.

4. (15 points) Let $g : \mathbb{R} \to \mathbb{Q}$ be a continuous map. Prove that g is a constant map, i.e., there exists a constant $\alpha \in \mathbb{Q}$ such that $g(x) = \alpha$ for all $x \in \mathbb{R}$.

5. (15 points) Let $f : \mathbb{Q} \to \mathbb{R}$ be a continuous map. Is it true that one can always find a continuous map $g : \mathbb{R} \to \mathbb{R}$ extending f, namely, $g(x) = f(x)$ for any $x \in \mathbb{Q}$? If true, prove it; if false, give a counter-example and prove that no such extension is possible.

6. (15 points) For each $N \in \mathbb{N}$, let $f_N(x) = \sum_{n=0}^{N} 3^{-n} \sin(2^n x + n)$ be a real valued function on $\mathbb{R}$. Prove that as $N \to \infty$, f_N converges uniformly to some real valued continuous function.

1. (1) The set $(-1,1) \cap \mathbb{Q}$ is open in $\mathbb{Q}$, by definition of the induced topology. Or directly, we may check that $\forall\, \alpha \in (-1,1) \cap \mathbb{Q}$, we may choose $\delta = \min\{\alpha-1, 1-\alpha\}$. and $B_\delta(\alpha) \subset (-1,1) \cap \mathbb{Q}$.

It is not closed in $\mathbb{Q}$, since its closure in $\mathbb{Q}$ includes the points $\{-1\}$. and $\{1\}$. For example, the sequence $(1-\frac{1}{n})_{n\in\mathbb{N}}$ is a sequence in this set, and converges to $1 \in \mathbb{Q}$, but 1 is not in this set.

(2). $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$. Hence, by the induced topology on $\mathbb{Q}$, it is both open and closed.

(3). $(0,2) \cap \mathbb{Q}$ is not connected. In fact, we will prove that any subset $S \subset \mathbb{Q}$ consisting of more than 1 elements is disconnected. Suppose $\alpha, \beta \in S$., and $\alpha < \beta$, then we may take any $\gamma \in \mathbb{R} \setminus \mathbb{Q}$, s.t. $\alpha < \gamma < \beta$., and consider $S_1 = S \cap (-\infty, \gamma) = S \cap (-\infty, \gamma]$. and $S_2 = S \cap (\gamma, +\infty) = S \cap [\gamma, +\infty)$. Then S_1 and S_2 are both open in S, and non-empty ($\alpha \in S_1$ and $\beta \in S_2$), and $S = S_1 \cup S_2$, hence S is disconnected.

(4). $K = [0,1] \cap \mathbb{Q}$. It is closed and bounded in $\mathbb{Q}$. But K is not compact, since as a subset of $\mathbb{R}$, K

is not closed, hence K is not compact as a subset of $\mathbb{R}$. Since the notion of compactness does not depends on the ambient space, we say K is not compact.

More concretely, we may pick a sequence (x_n) in $[0,1] \cap \mathbb{Q}$ convergent in $\mathbb{R}$ to some irrational number α. This sequence. does not have a convergent subsequence in $\mathbb{Q}$, hence K is not sequentially compact.

(5) $K = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is compact (as subset in $\mathbb{R}$, hence also as subset in $\mathbb{Q}$).

2. as above.

For statement about bounded set, consider

$$f : (0,\infty) \to (0,\infty) \qquad x \mapsto \frac{1}{x}.$$

then $f((0,1)) = (1,\infty)$ image of bounded set ~~is~~ can be unbounded.

$f^{-1}((0,1)) = (1,\infty)$ image of bounded set can be unbounded.

Pf:

3. Let $\{U_\alpha\}_{\alpha\in A}$ be a covering of $K_1 \cup \cdots \cup K_n$. then it is also a covering for each K_i. By compactness of K_i, we can find finite subsets $A_i \subset A$, such that $K_i \subset \bigcup_{\alpha \in A_i} U_\alpha$. Let $\tilde{A} = A_1 \cup \cdots \cup A_n$, then $\tilde{A}$ is

a finite subset, and $K = \bigcup_{i=1}^{n} K_i \subset \bigcup_{i=1}^{n} \left(\bigcup_{\alpha \in A_i} U_\alpha \right)$

$$= \bigcup_{\alpha \in \tilde{A}} U_\alpha.$$

Hence K is compact.

4. Pf: Since $\mathbb{R}$ is connected, f is continuous, hence $f(\mathbb{R})$ is connected in $\mathbb{Q}$.

Suffice to prove that the only connected subsets in $\mathbb{Q}$ are of the form $\{\alpha\}$. This was proven in problem 1.3.

5. No. • Take $f: \mathbb{Q} \to \mathbb{R}$ as following:

$$f(x) = \begin{cases} -1 & x < \sqrt{2},\ x \in \mathbb{Q} \\ 1 & x > \sqrt{2},\ x \in \mathbb{Q}. \end{cases}$$

To see f is continuous, we only need to check $\mathbb{Q}$, ϕ, $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ and $(\sqrt{2}, +\infty) \cap \mathbb{Q}$ are open subsets in $\mathbb{Q}$. since these are the only possible preimages $f^{-1}(E)$ for $E \subset \mathbb{R}$.

• Suppose there exists a continuous extension g to $\mathbb{R}$, then the left limit at $\sqrt{2}$ and right limit at $\sqrt{2}$ would be

$$g(\sqrt{2}+) = f(\sqrt{2}+) = 1,$$
$$g(\sqrt{2}-) = f(\sqrt{2}-) = -1.$$

contradicting with the requirement that $g(\sqrt{2}-) = g(\sqrt{2}+)$.
Hence, no continuous extension exists.

6. This holds by Weierstrass M-test. since

the summand $\left| 3^{-n} \sin(n^2 x + n) \right| \leq 3^{-n}$. and
$\sum_{n=0}^{\infty} 3^{-n} < \infty$.

And uniformly limit of continuous function