1. Show that if f is integrable on $[a,b]$, then for any sub interval $[c,d]$, f is also integrable.

Pf: Suffice to show that, for any $\varepsilon>0$, there exists a partition P of $[c,d]$, s.t.

$$\mathcal{U}(P,f,\alpha) - L(P,f,\alpha) < \varepsilon. \qquad (*)$$

Since f is integrable on $[a,b]$, then there exists a partition Q of $[a,b]$, s.t.

$$\mathcal{U}(Q,f,\alpha) - L(Q,f,\alpha) < \varepsilon. \qquad (**)$$

We may refine Q so that points c, d are in $Q.$, and $(**)$ also holds under this refinement. Thus

$$Q = \{a = x_0 < x_1 < x_2 < \cdots < x_k = c < x_{k+1} < \cdots < x_m = d < \cdots < x_n = b\}$$

then we may let $P = \{x_k < x_{k+1} < \cdots < x_m\}.$, the subset of partition Q on $[c,d]$. Thus.

$$\mathcal{U}(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=k+1}^{m} (M_i - m_i)[\alpha(x_i) - \alpha(x_{i-1})]$$

$$\leq \sum_{i=1}^{n} (M_i - m_i)(\alpha(x_i) - \alpha(x_{i-1}))$$

$$= \mathcal{U}(Q,f,\alpha) - L(Q,f,\alpha). < \varepsilon$$

Thus we have constructed a partition P of $[c,d]$ satisfying $(*)$.

(2). Suppose f is a continuous non-negative function on $[a,b]$. Show that $\int_a^b f(x)\,dx = 0 \Rightarrow f(x) = 0 \quad \forall x \in [a,b]$

Pf: suffice $\exists\, x_0 \in [a,b]$, that $f(x_0) > 0$, then.
Let $\varepsilon = \frac{1}{2} f(x_0)$, $\exists\, \delta > 0$. s.t. $\forall\, x \in [a,b]$, $|x - x_0| < \delta$,
$|f(x) - f(x_0)| < \varepsilon$. Thus $f(x) > f(x_0) - \varepsilon = \frac{1}{2} f(x_0)$,
$\forall\, x \in B_\delta(x_0) \cap [a,b]$. Let $[c,d] \subset B_\delta(x_0) \cap [a,b]$, $d > c$. Then

$$\int_a^b f(x)\, dx \geqslant \int_c^d f(x)\, dx \geqslant \int_c^d \varepsilon \cdot dx = \varepsilon \cdot (d-c) > 0$$

contradict with $\int_a^b f\, dx = 0$

3. Let $f(x)$ be a function on $[0,1]$, with.

$$f(x) = \begin{cases} 0 & x = 0 \\ \sin\left(\frac{1}{x}\right) & x > 0 \end{cases}$$

and let α be given by

$$\alpha(x) = \begin{cases} 0 & x = 0 \\ \sum\limits_{\frac{1}{n} < x} 2^{-n} & x > 0. \end{cases}$$

Show that $\int_0^1 f(x)\, d\alpha(x)$ exists.

Pf: Since $\sum_{n=1}^{\infty} 2^{-n} < \infty$, hence as $N \to \infty$

$\sum_{n=N}^{\infty} 2^{-n} \to 0$. Thus, as $x \to 0$, $\frac{1}{x} \to \infty$

hence $\alpha(x) = \sum\limits_{n > \frac{1}{x}} 2^{-n} \to 0$. Thus $\alpha(x)$ is continuous

at $x = 0$. Since $f(x)$ is a bounded real function with discontinuity only at $x=0$, and α is continuous at $x=0$, Hence by Thm 6.10 in Rudin, $f \in R(\alpha)$ #

#4. If $p, q > 0$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then if f, g are Riemann integrable over $[a,b]$, then

$$\left| \int_a^b fg \, dx \right| \leq \left[\int_a^b |f|^p dx \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g|^q dx \right]^{\frac{1}{q}}.$$

Pf: (a) We first show that $\forall\, u, v \geq 0$,

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

The case is clear if $u=0$ or $v=0$. Hence we only consider the case that $u>0$ and $v>0$. For fixed p, q, u, we consider the function $\varphi(v) = \frac{v^q}{q} + \frac{u^p}{p} - uv$. Then since $q > 1$, $\lim_{v\to\infty} \varphi(v) = +\infty$. And $\varphi'(v) = v^{q-1} - u$, hence as $v \to 0$.

$$\lim_{v\to 0} \varphi'(v) = -u < 0, \text{ and } \lim_{v \to 0} \varphi(v) = \varphi(0) = \frac{u^p}{p} > 0.$$

Hence the $\inf \{\varphi(v) \mid v > 0\}$ is achieved for some $v_0 \in (0,\infty)$, and at $v = v_0$, $\varphi'(v_0) = 0$. Thus $v_0^{q-1} = u \Leftrightarrow v_0 = u^{\frac{1}{q-1}}$.

Note that $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{q}{p} + 1 = q$, $\Leftrightarrow \frac{q}{p} = q-1 \Leftrightarrow \frac{1}{q-1} = \frac{p}{q}$.

Thus v_0 satisfies $v_0^q = u^p$,

$$\varphi(v_0) = \frac{u^p}{p} + \frac{u^p}{q} - u \cdot u^{\frac{p}{q}} = u^p - u^{1+\frac{p}{q}} = u^p - u^p = 0.$$

Hence $\forall\, v \in (0,\infty)$, $\varphi(v) \geq \varphi(v_0) = 0$.

(b). Suppose f, g are non-negative functions, and satisfies $\int f^p \, dx = 1$, $\int g^q \, dx = 1$, then.

$$\int_a^b f(x)\, g(x)\, dx \leq \int_a^b \frac{f(x)^p}{p} + \frac{g(x)^q}{q} \, dx = \frac{1}{p} + \frac{1}{q} = 1.$$

(c) For general integrable function f, g, we have

$$\int_a^b f \cdot g \cdot dx \leq \left| \int_a^b f \cdot g \, dx \right| \leq \int_a^b |f| \cdot |g| \, dx$$

If $|f| \equiv 0$ or $|g| \equiv 0$ for all $x \in [a,b]$, then there is nothing to prove, hence we may assume $|f|$ and $|g|$ are not identically zero. Let

$$A = \left[\int_a^b |f|^p \, dx \right]^{\frac{1}{p}} \quad B = \left[\int_a^b |g|^q \cdot dx \right]^{\frac{1}{q}}$$

then $A, B > 0$. Let

$$F(x) = |f(x)| / A, \qquad G(x) = |g(x)| / B,$$

then $\int F(x)^p \, dx = 1$, $\int G(x)^q dx = 1$. Hence by (b),

$$\int F(x) \cdot G(x) \cdot dx \leq 1.$$

Thus

$$\int_a^b |f(x)| \cdot |g(x)| \, dx \leq A \cdot B = \left[\int_a^b |f|^p \, dx \right]^{\frac{1}{p}} \left[\int_a^b |g(x)|^q \cdot dx \right]^{\frac{1}{q}}.$$

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