Rudin Ch 4 #14. 20. 21. 25(a).

#14 Let $f: [0,1] \to [0,1]$ continuous map, show that

$\exists x \in [0,1]$, s.t. $f(x) = x$.

Pf: If $f(0) = 0$ or $f(1) = 1$. then we are done.

Assume then $f(0) > 0$ and $f(1) < 1$. Let $g(x) = x - f(x)$.

then $g(0) < 0$ and $g(1) > 0$. Hence by intermediate value thm,

$\exists x \in (0,1)$, s.t. $g(x) = 0$. ie. $f(x) = x$.

#20 Let $E \subset X$ a subset of metric space X. Define

$$\rho_E(x) = \inf_{y \in E} \{d(x,y)\}.$$

(a) show that $\rho_E(x) = 0$ iff $x \in \overline{E}$

(b). show that $\rho_E(x)$ is uniformly continuous on X.

Pf: (a) $\rho_E(x) = 0 \iff \forall n \in \mathbb{N}, \exists x_n \in E$, s.t. $d(x, x_n) < \frac{1}{n}$

$\iff \exists$ a seq (x_n) in E, $\lim_n x_n = x$

$\iff x \in \overline{E}$

(b). $\forall \varepsilon > 0$, we need to find $\delta > 0$, s.t. if $d(x,y) < \delta$,

then $|\rho_E(x) - \rho_E(y)| \leq \varepsilon$. Let $\delta = \varepsilon$. Then.

$$\rho_E(x) = \inf_{z \in E} d(x,z) \leq \inf_{z \in E} d(y,z) + d(x,y) = \delta + \rho_E(y) = \varepsilon + \rho_E(y)$$

$\therefore \quad \rho_E(x) \leq \rho_E(y) + \varepsilon$ similarly, $\rho_E(y) \leq \rho_E(x) + \varepsilon$.

Hence $\quad |\rho_E(x) - \rho_E(y)| \leq \varepsilon$

#21 Suppose K and F are disjoint sets in X. K compact. F is closed. Prove that $\exists\, \delta > 0$. s.t.

$$d(p,q) > \delta \quad \forall\, p \in K,\ q \in F$$

If K and F are only closed, show that $\inf \{d(p,q) \mid p \in K, q \in F\}$ can be 0.

Pf: The distance function to F, ρ_F is continuous. If $x \notin F$, then $\rho_F(x) > 0$ by previous problem. Hence $\rho_F(x)$ on K is positive. Since K is compact., $\inf\limits_{x \in K} \rho_F(x) = \rho_F(p)$ for some point $p \in K$., hence $\inf\limits_{x \in K} \rho_F(x) > 0$. Let $\delta = \frac{1}{2} \inf\limits_{x \in K} \rho_F(x)$ will do.

with induced metric...

Let $X = \mathbb{R} \setminus \{0\}$! $K = [-1, 0)$ and $F = (0, 1]$. then K and F are compact Then $x_n = -\frac{1}{n} \in K$. $y_n = \frac{1}{n} \in F$, $d(x_n, y_n) = \frac{2}{n} \to 0$.

#25(a) If $K \subset \mathbb{R}^n$ is compact and $C \subset \mathbb{R}^n$ is closed, prove that $K + C$ is closed.

Pf: (a). Let $x \notin K + F$, we need to show that $\exists\, \delta > 0$. s.t. $B_\delta(x) \cap K+F = \phi$.

$x \notin K+F \iff x-K \cap F = \phi.$

Hence, by Prob#21, $\exists\, \delta > 0$, s.t. $\forall\, y \in x-K$, $z \in F$.

$d(y,z) > \delta$. Thus. $B_\delta(y) \cap F = \phi$., $\forall\, y \in x-K$.

$\therefore\quad B_\delta(x) - K \cap F = \phi$

$\Rightarrow\quad B_\delta(x) \cap K+F = \phi$

$\Rightarrow\quad K+F$ is closed.

<u>alternative proof</u>: we only to to show that, if $p_n \in K+F$ converges to p in $\mathbb{R}^n$, then $p \in K+F$.

For each p_n, we write $p_n = x_n + y_n$, $x_n \in K$, $y_n \in F$.

Then, by passing to a subseq, we may assume $x_n \to x$ in K.

Then, since $y_n = p_n - x_n$, and $p_n \to p$, $x_n \to x$, then hence y_n also converges to $y = p - x$. Since F is closed, $y \in F$. Thus, $p = x+y \in K+F$. #.