Math 104. HW 7. Rudin 1, 2, 4, 6, 7, 18. Ch4.

1. Suppose f is a real valued function on $\mathbb{R}$, which satisfies.

$$\lim_{h \to 0} f(x+h) - f(x-h) = 0.$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

Sol'n: No. For example $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$, this function is not continuous at $x = 0$.

In general, if $E \subset \mathbb{R}$ is a discrete set, i.e. the limit set E' is empty, then for any function $g: E \to \mathbb{R} \setminus \{0\}$ function

$$f(x) = \begin{cases} g(x) & x \in E \\ 0 & x \notin E \end{cases}$$

is not continuous, and satisfies $\lim_{h \to 0} f(x+h) - f(x-h) = 0$.

2. If $f: X \to Y$ is a continuous mapping from a metric space X to a metric space Y, prove that

$$f(\bar{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. Show that, by an example, that $f(\bar{E})$ can be a proper subset.

Pf: For any $p \in \bar{E}$, there exists a seq of points (p_n) with $p_n \in E$, such that $\lim_n p_n = p$. Hence, by continuity of f.

$f(p) = \lim_n f(p_n) \in \overline{f(E)}$. #

Pf #2: $f(\overline{E}) \subset \overline{f(E)} \iff \overline{E} \subset f^{-1}(\overline{f(E)})$

$\iff f^{-1}(\overline{f(E)})$ is closed and contains E. (*)

· Since f is continuous, and $\overline{f(E)}$ is closed.

hence $f^{-1}(\overline{f(E)})$ is closed.

· furthermore, $f^{-1}(\overline{f(E)}) \supset f^{-1}(f(E)) \supset E$.

Hence, condition (*) is satisfied.

Example that the inclusion is proper:

$X = (0,1)$, $Y = \mathbb{R}$, $f: X \to Y$ the inclusion map.

take $E = (0,1)$., then the closure of E in X is E itself.

The closure of $f(E)$ in Y is $[0,1]$.

Here is another example:

$f: \mathbb{R} \to \mathbb{R}$. $f(x) = e^x$.

Let $E = \mathbb{R}$, then $f(\mathbb{R}) = (0,\infty)$.

$f(\overline{E}) = f(\mathbb{R}) = (0,\infty)$. , $\overline{f(\mathbb{R})} = [0,\infty)$.

#4 Let f, g be continuous map from X to Y. Suppose there is a dense subset $E \subset X$, s.t. $f|_E = g|_E$.

Show that $f = g$.

Pf: For any $x \in X$, there exist a seq. (x_n) in E, such that $x_n \to x$. By continuity of f and g, and $f|_E = g|_E$, we have. $f(x) = \lim f(x_n) = \lim g(x_n) = g(x)$.. #

#6: Suppose $f: E \to Y$ is a map, and let

$$\Gamma_f = \{ (x, f(x)) \mid x \in E \} \subset E \times Y$$

be the graph of f. Suppose E is compact. Then;

$$f \text{ is continuous} \Leftrightarrow \Gamma_f \text{ is compact.}$$

We give 2 proofs.

Pf #1. $\Rightarrow$ let $F: E \to E \times Y$ be given by $x \mapsto (x, f(x))$. then Γ_f is the image of $F(E)$. Since f is continuous, hence F is continuous. Since E is compact and F is continuous, hence $F(E)$ is compact. That is, Γ_f is compact.

$\Leftarrow$ We need to prove that, for any open subset $V \subset Y$, the preimaage $f^{-1}(V)$ is open. Or equivalent, for any closed subset $K \subset Y$, the preimage $f^{-1}(K)$ is closed. Consider the projection maps $\pi_Y: E \times Y \to Y$, $\pi_E: E \times Y \to E$, they are all continuous. Then

$$f^{-1}(K) = \pi_E \left(\Gamma_f \cap \pi_Y^{-1}(K) \right)$$

We see $\pi_Y^{-1}(K)$ is closed, and Γ_f is compact, hence $\pi_Y^{-1}(K) \cap \Gamma_f$ is compact, and its image under π_E is also compact. Thus $f^{-1}(K)$ is compact, hence closed.

<u>Pf #2:</u> $\Rightarrow$ Let $(x_n, y_n) \in \Gamma_f$ be a seq of points, $n \in \mathbb{N}$. we need to show that there is a convergent subseq. Since (x_n) is a seq of points in E, ~~too~~ and E is compact, we have a subseq $(x_{n_k})_k \to x_0$. By continuity of f, $f(x_{n_k}) \to f(x_0)$. I.e. $(y_{n_k}) \longrightarrow f(x_0)$. Hence, $(x_{n_k}, y_{n_k}) \to (x_0, f(x_0))$ is a convergent subseq.

$\Leftarrow$ We need to prove that, for any sequence $x_n \to x$ in E, we have $f(x_n) \to f(x)$. Suppose not, then $\exists\, \varepsilon > 0$. s.t. we have a subsequence (x_{n_k}), s.t. $|f(x_{n_k}) - f(x)| > \varepsilon$. Replace (x_n) by (x_{n_k}), we may assume that $|f(x_n) - f(x)| > \varepsilon$ $\forall n$. Then consider the seq in the graph $\{(x_n, f(x_n))\}$. By compactness of Γ_f, there is a subseq such that $(x_{n_k}, f(x_{n_k})) \to (x_0, y_0)$. Hence,
$\lim_k x_{n_k} = x_0$, and $\lim_k f(x_{n_k}) = y_0$. Since (x_{n_k}) is a subseq of (x_n), and (x_n) converges to x, we have $x_0 = x$. Since $|f(x_{n_k}) - f(x)| > \varepsilon$, $\forall k$, we have $|y_0 - f(x)| \geq \varepsilon$. However, since $(x_0, y_0) \in \Gamma_f$, hence $y_0 = f(x_0) = f(x)$. Thus, we have a contradiction.

#7. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \end{cases} \qquad g(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{xy^2}{x^2+y^6} & \text{else} \end{cases}$$

Prove that :
- f is bounded on $\mathbb{R}^2$
- g is unbounded on every nbhd of $(0,0)$
- f is not continuous at $(0,0)$.
- f, g are continuous on every line through origin.

Pf : • for $a \geq 0$, $b \geq 0$, we have $a^2 + b^2 \geq 2ab$,

hence $|xy^2| \leq \frac{1}{2}(|x|^2 + |y|^4)$. Thus, if $(x,y) \neq (0,0)$,

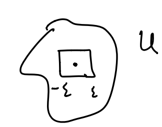
then $|f(x,y)| \leq \dfrac{\frac{1}{2}(|x|^2 + |y|^4)}{x^2+y^4} \leq \frac{1}{2}$. (note that $x^2 \geq 0$, $y^4 \geq 0$, $x^2+y^4 > 0$.)

and $f(0,0) = 0$. Hence

$$|f(x,y)| \leq \frac{1}{2} \qquad \forall (x,y) \in \mathbb{R}^2$$

• For any nbhd U of $(0,0)$, there exists an $1 > \varepsilon > 0$, such that $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \subset U$. Hence, for small enough t, e.g. $|t| < \varepsilon$, we have $|t|^3 < \varepsilon$, and

$$(t^3, t) \in (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \subset U.$$



If $0 < |t| < \varepsilon$, then

$$g(t^3, t) = \frac{t^3 \cdot t^2}{(t^3)^2 + t^6} = \frac{t^5}{2t^6} = \frac{1}{2t}.$$

Hence, as $t \to 0+$, $g(t^3, t) \to \infty$.

• Consider a sequence of points (x,y), that converges to $(0,0)$, such that the ratio $\frac{x^2}{y^4}$ is constant. e.g. $(x_n, y_n) = \left(\frac{c}{n^2}, \frac{1}{n}\right)$

then, $f(x_n, y_n) = \dfrac{(\frac{c}{n^2})\cdot(\frac{1}{n})^2}{(\frac{c}{n^2})^2 + (\frac{1}{n})^4} = \dfrac{c}{1+c^2}$. Hence, for different value of c, the limit $\lim_{n\to\infty} f(x_n, y_n)$ is different. Thus, $f(x,y)$ is not continuous at $(0,0)$.

- If the line is the y-axis, i.e. $x=0$, then.

$$f(0,y) = \frac{0}{0+y^4} = 0. \qquad y \neq 0.$$

$$f(0,0) = 0.$$

Hence $f(0,y)$ is continuous.

If the line is of slope $y = kx$, then, we have

$$f(x, kx) = \frac{x\cdot(kx)^2}{x^2+(kx)^4} = \frac{kx}{1+k^4\cdot x^2}$$

$$\lim_{x\to 0} f(x,kx) = \frac{\lim_{x\to 0} kx}{\lim (1+k^4x^2)} = \frac{0}{1} = 0. = f(0,0)$$

Hence, the restriction of $f(x,y)$ on the line $y=kx$ is continuous.

The discuss for $g(x,y)$ is similar.

<u>#18</u> Every rational x can be written uniquely as m/n, where, $m, n \in \mathbb{Z}$. $n>0$, and m. n are coprime. When $x=0$, take $n=1$. Consider the function

$$f(x) = \begin{cases} 1/n & x = m/n \\ 0 & x \text{ irrational}. \end{cases}$$

show that f is continuous at every irrational point.

<u>Pf</u> : We claim that, for any $x_0 \in \mathbb{R}$, $\lim\limits_{\substack{x \to x_0 \\ x \neq x_0}} f(x) = 0$.
We need to show that, for any $\varepsilon > 0$, $\exists\ \delta > 0$. s.t. $\forall\ 0 < |y - x| < \delta$, $|f(y)| < \varepsilon$. Let $N > 0$ be an integer, large enough, such that $\frac{1}{N} < \varepsilon$. Then consider the subset

$$A_N = \mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \cdots \cup \frac{1}{N}\mathbb{Z}.$$

where $\frac{1}{n}\mathbb{Z} = \{ m/n \mid m \in \mathbb{Z} \}$. Each $\frac{1}{n}\mathbb{Z}$ is a closed subset, hence A_N is a closed subset. If $y \in \mathbb{R} \setminus A_N$, then $0 \leq f(y) < \frac{1}{N}$. If $x_0 \in \mathbb{R} \setminus A_N$, then since $\mathbb{R} \setminus A_N$ is open, we have $\delta > 0$ small enough, such that $(x_0 - \delta, x_0 + \delta) \subset \mathbb{R} \setminus A_N$. This is the desired δ. If $x_0 \in A_N$, then we claim that $\forall x \in A_N \setminus \{x_0\}$, $|x_0 - x| > \frac{1}{N^2}$. Indeed, $\left| \frac{m_1}{n_1} - \frac{m_2}{n_2} \right| = \frac{|m_1 n_2 - m_2 n_1|}{|n_1 n_2|} \geq \frac{1}{n_1 n_2} \geq \frac{1}{N^2}$.
Hence, we may take $\delta = \frac{1}{N^2}$.

Note that, for the first part of #18, since x_0 is irrational, hence the case $x_0 \in A_N$ will not happen. And student may safely ignore this case.